Brownian limits, local limits, extreme value and variance asymptotics for convex hulls in the ball

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Abstract

The paper [40] establishes an asymptotic representation for random convex polytope geometry in the unit ball $\mathbb{B}_d,\ d \geq 2$, in terms of the general theory of stabilizing functionals of Poisson point processes as well as in terms of the so-called *generalized paraboloid growth process*. This paper further exploits this connection, introducing also a dual object termed the *paraboloid hull process*. Via these growth processes we establish local functional and measure-level limit theorems for the properly scaled radius-vector and support functions as well as for curvature measures and k-face empirical measures of convex polytopes generated by high density Poisson samples. We use general techniques of stabilization theory to establish Brownian sheet limits for the defect volume and mean width functionals, and we provide explicit variance asymptotics and central limit theorems for the k-face and intrinsic volume functionals. We establish extreme value theorems for radius-vector and support functions of random polytopes and we also establish versions of the afore-mentioned results for large isotropic cells of hyperplane tessellations, reducing the study of their asymptotic geometry to that of convex polytopes via inversion-based duality relations [14].

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1 Introduction

Let K be a smooth convex set in $\mathbb{R}^d$ of unit volume. Letting $\mathcal{P}_\lambda$ be a Poisson point process in $\mathbb{R}^d$ of intensity λ we let K_λ be the convex hull of $K \cap \mathcal{P}_\lambda$. The random polytope K_λ, together with the analogous polytope K_n obtained by considering n i.i.d. uniformly distributed points in K, are well-studied objects in stochastic geometry.

The study of the asymptotic behavior of the polytopes K_λ and K_n, as $\lambda \to \infty$ and $n \to \infty$ respectively, has a long history originating with the work of Rényi and Sulanke [31]. The following functionals of K_λ have featured prominently:

- The volume $\mathrm{Vol}(K_\lambda)$ of K_λ, abbreviated as $V(K_\lambda)$,
- The number of k-dimensional faces of K_λ, denoted $f_k(K_\lambda),\ k \in \{0, 1, ..., d-1\}$; in particular $f_0(K_\lambda)$ is the number of vertices of K_λ,
- The mean width $W(K_\lambda)$ of K_λ,
- The distance between ∂K_λ and ∂K in the direction $u \in \mathbb{R}^d$, here denoted $r_\lambda(u)$,
- The distance between the boundary of the Voronoi flower defined by $\mathcal{P}_\lambda$ and ∂K in the direction $u \in \mathbb{R}^d$, here denoted $s_\lambda(u)$,
- The k-th intrinsic volumes of K_λ, here denoted $V_k(K_\lambda),\ k \in \{1, ..., d-1\}$.

The mean values of these functionals, as well as their counterparts for K_n, are well-studied and for a complete account we refer to the surveys of Affentranger [1], Buchta [12], Gruber [17], Schneider [35, 37], and Weil and Wieacker [44]), together with Chapter 8.2 in Schneider and Weil [38]. There has been recent progress in establishing higher order and asymptotic normality results for these functionals, for various choices of K. We signal the important breakthroughs by Reitzner [33], Bárány and Reitzner [4], Bárány et al. [3], and Vu [45, 46]. These results, together with those of Schreiber and Yukich [40], are difficult and technical, with proofs relying upon tools from convex geometry and probability, including martingales, concentration inequalities, and Stein's method. When K is the unit radius d-dimensional ball $\mathbb{B}_d$ centered at the origin, Schreiber and Yukich [40] establish variance asymptotics for $f_0(K_\lambda)$, but, up to now, little is known regarding explicit variance asymptotics for other functionals of K_λ.

This paper has the following goals. We first study two processes in *formal space-time* $\mathbb{R}^{d-1} \times \mathbb{R}_+$, one termed the *parabolic growth process* and denoted by Ψ, and a *dual process termed the parabolic hull process*, here denoted by Φ. While the first process was introduced in [40], the second has apparently not been considered before. When $K = \mathbb{B}_d$, an embedding of convex sets into the space of continuous functions on the unit sphere $\mathbb{S}_{d-1}$, together with a re-scaling, show that these processes are naturally suited to the study of K_λ. The spatial localization can be exploited to describe first and second order asymptotics of functionals of K_λ. Many of our main results, described as follows, are obtained via consideration of the processes Ψ and Φ. Our goals are as follows:

- Show that the distance between K_λ and $\partial\mathbb{B}_d$, upon proper re-scaling in a local regime, converges in law as $\lambda \to \infty$, to a continuous path stochastic process defined in terms of Φ, adding to Molchanov [24]; similarly, we show that the distance between $\partial\mathbb{B}_d$ and the Voronoi flower defined by $\mathcal{P}_\lambda$ converges in law to to a continuous path stochastic process defined in terms of Ψ. In the two-dimensional case the fidis (finite-dimensional distributions) of these distances, when properly scaled, are shown to converge to the fidis of Ψ and Φ, whose description is given explicitly, adding to work of Hsing [19].

- Show, upon properly re-scaling in a global regime, that the suitably integrated local defect width and defect volume functionals, when considered as processes indexed by points in $\mathbb{R}^{d-1}$ mapped on $\partial\mathbb{B}_d$ via the exponential map, satisfy a functional central limit theorem, that is converge in the space of continuous functions on $\mathbb{R}^{d-1}$ to a Brownian sheet on the injectivity region of the exponential map, with respective explicit variance coefficients σ_W^2 and σ_V^2 given in terms of the processes Ψ and Φ. To the best of our knowledge, this connection between the geometry of random polytopes and Brownian sheets is new. In particular
$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \mathrm{Var}[W(K_\lambda)] = \sigma_W^2, \tag{1.1}$$
and
$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \mathrm{Var}[V(K_\lambda)] = \sigma_V^2. \tag{1.2}$$
This adds to Reitzner's central limit theorem (Theorem 1 of [33]), his variance approximation $\mathrm{Var}[V(K_\lambda)] \approx \lambda^{-(d+3)/(d+1)}$ (Theorem 3 and Lemma 1 of [33]), and Hsing [19], which is confined to $d = 2$.

- Establish central limit theorems and variance asymptotics for the number of k-dimensional faces of K_λ, showing for all $k \in \{0, 1, ..., d-1\}$

$$\lim_{\lambda \to \infty} \lambda^{-(d-1)/(d+1)} \mathrm{Var}[f_k(K_\lambda)] = \sigma_{f_k}^2, \tag{1.3}$$

  where $\sigma_{f_k}^2$ is described in terms of the processes Ψ and Φ. This improves upon Reitzner (Lemma 2 of [33]), whose breakthrough paper showed $\mathrm{Var}[f_k(K_\lambda)] \approx \lambda^{(d-1)/(d+1)}$, and builds upon [40], which establishes (1.3) only for $k = 0$.

- Establish central limit theorem and variance asymptotics for the intrinsic volumes $V_k(K_\lambda)$ for all $k \in \{1, ..., d-1\}$, namely

$$\lim_{\lambda \to \infty} \lambda^{(d+3)/(d+1)} \mathrm{Var}[V_k(K_\lambda)] = \sigma_{V_k}^2, \tag{1.4}$$

  where again $\sigma_{V_k}^2$ is described in terms of the processes Ψ and Φ. This adds to Bárány et al. (Theorem 1 of [3]), which shows $\mathrm{Var}[V_k(K_n)] \approx n^{-(d+3)/(d+1)}$.

- Show that the distribution function of extremal values of these distances converges to Gumbel type extreme distributions, which extends to some extent the two-dimensional results due to Bräker et al. [10].

- Extend some of the preceding results to large zero-cells of certain isotropic hyperplane tessellations. We use the duality relation introduced in [14] to investigate the geometry of the cell containing the origin, under the condition that it has large inradius. This adds to previous works [20] connected to D. G. Kendall's conjecture.

The limits (1.1)-(1.4) resolve the issue of finding explicit variance asymptotics for face functionals and intrinsic volumes, a long-standing problem put forth this way in the 1993 survey of Weil and Wieacker (p. 1431 of [44]): 'We finally emphasize that the results described so far give mean values hence first-order information on random sets and point processes... There are also some less geometric methods to obtain higher-order informations or distributions, but generally the determination of the variance, e.g., is a major open problem'.

These goals are stated in relatively simple terms and yet they and the methods behind them suggest further objectives involving additional explanation. One of our chief objectives is to carefully define the growth processes Ψ and Φ and exhibit their geometric properties making them relevant to K_λ, including their crucial localization in space, known as *stabilization*. The latter property provides the key towards establishing variance asymptotics and the limit theory of functionals of K_λ. A second objective is to describe two natural scaling regimes, one suited for locally defined functionals of K_λ, and the other suited for the integrated characteristics of K_λ, namely the width and volume functionals. A third objective is to extend the results indicated above to ones holding on the level of measures. In other words, functionals considered here are naturally associated with random measures such as the k-face empirical measures and certain integral geometric measures related to intrinsic volumes; we will show variance asymptotics for such measures and also convergence of their fidis to those of a Gaussian process under suitable global scaling. Further, these random point measures, though defined on finite volume regions, are shown to converge in law under a local scaling regime to limit measures on infinite volume *formal space-time* half-spaces, also explicitly described in terms of the processes Ψ and Φ. We originally intended to restrict attention to convex hulls generated from Poisson points with intensity density λ, but realized that the methods easily extend to treat intensity densities decaying as a power of the distance to the boundary of the unit ball as given by (2.1) below, and so we treat this more general case without further complication. The final major objective is to study the extreme value behavior of the polytope K_λ. These major objectives are discussed further in the next section. We expect that much of the limit theory described here can be 'de-Poissonized', that is to say extends to functionals of the polytope K_n. Finally, we also expect that the duality between convex hulls and zero-cells of Poisson hyperplane tessellations should imply additional results for the latter, notably asymptotics for intrinsic volumes and curvature measures. We leave these issues for later investigation.

2 Basic functionals, measures, and their scaled versions

Basic functionals and measures. Given a locally finite subset $\mathcal{X}$ of $\mathbb{R}^d$ we denote by $\text{conv}(\mathcal{X})$ the *convex hull* generated by $\mathcal{X}$. Given a compact convex $K \subset \mathbb{R}^d$ we let $h_K : \mathbb{S}_{d-1} := \partial\mathbb{B}_d \to \mathbb{R}$ be the support function of K, that is to say $h_K(u) := \sup\{\langle x, u\rangle : \ x \in K\}$. It is easily seen for $\mathcal{X} \subset \mathbb{R}^d$ and $u \in \mathbb{S}_{d-1}$ that

$$h_{\text{conv}(\mathcal{X})}(u) = \sup\{h_{\{x\}}(u) : \ x \in \mathcal{X}\} = \sup\{\langle x, u\rangle : \ x \in \mathcal{X}\}.$$

For $u \in \mathbb{S}_{d-1}$, the *radius-vector function* of K in the direction of u is given by $r_K(u) := \sup\{\varrho > 0, \ \varrho u \in K\}$. For $x \in \mathbb{R}^d$ denote by $p_K(x)$ the metric projection of x onto K, that is to say the unique point of K minimizing the distance to x. Clearly, if $x \notin K$ then $p_K(x) \in \partial K$. For $\lambda > 0$ we abuse notation and *henceforth denote by $\mathcal{P}_\lambda$ the Poisson point process in $\mathbb{B}_d$ of intensity*

$$\lambda(1-|x|)^\delta dx,\ x \in \mathbb{B}_d, \tag{2.1}$$

with some $\delta \geq 0$ to remain fixed throughout the paper. Further, abusing notation we put

$$K_\lambda := \operatorname{conv}(\mathcal{P}_\lambda).$$

The principal characteristics of K_λ studied here are the following functionals, the first two of which represent K_λ in terms of continuous functions on $\mathbb{S}_{d-1}$:

- The *defect support function.* For all $u \in \mathbb{S}_{d-1}$ we define

$$s_\lambda(u) := s(u, \mathcal{P}_\lambda), \tag{2.2}$$

  where for $\mathcal{X} \subseteq \mathbb{B}_d$ we define $s(u,\mathcal{X}) := 1 - h_{\operatorname{conv}(\mathcal{X})}(u)$. In other words, $s_\lambda(u)$ is the defect support function of K_λ in the direction u. It is easily verified that $s(u,\mathcal{X})$ is the distance in the direction u between the sphere $\mathbb{S}_{d-1}$ and the *Voronoi flower*

$$F(\mathcal{X}) := \bigcup_{x\in\mathcal{X}} B_d(x/2, |x|/2) \tag{2.3}$$

  where here and henceforth $B_d(y,r)$ denotes the d-dimensional radius r ball centered at y.

- The *defect radius-vector function.* For all $u \in \mathbb{S}_{d-1}$ we define

$$r_\lambda(u) := r(u, \mathcal{P}_\lambda), \tag{2.4}$$

  where for $\mathcal{X} \subseteq \mathbb{B}_d$ and $u \in \mathbb{S}_{d-1}$ we put $r(u,\mathcal{X}) := 1 - r_{\operatorname{conv}(\mathcal{X})}(u)$. Thus, $r_\lambda(u)$ is the distance in the direction u between the boundary $\mathbb{S}_{d-1}$ and K_λ. The convex hull K_λ contains the origin except on a set of exponentially small probability as $\lambda \to \infty$, and thus for asymptotic purposes *we assume without loss of generality that K_λ always contains the origin* and therefore the radius vector function $r_\lambda(u)$ is well-defined.

- The *numbers of k-faces.* Let $f_{k;\lambda} := f_k(K_\lambda)$, $k \in \{0,1,,...,d-1\}$, be the number of k-dimensional faces of K_λ. In particular, $f_{0;\lambda}$ and $f_{1;\lambda}$ are the number of vertices and edges, respectively. The spatial distribution of k-faces is captured by the k-face empirical measure (point process) $\mu_\lambda^{f_k}$ on $\mathbb{B}_d$ given by

$$\mu_\lambda^{f_k} := \sum_{f\in\mathcal{F}_k(K_\lambda)} \delta_{\operatorname{Top}(f)} \tag{2.5}$$

where $\mathcal{F}_k(K_\lambda)$ *is the collection of all k-faces of* K_λ and, for all $f \in \mathcal{F}_k(K_\lambda)$, $\mathrm{Top}(f)$ is the point in $\bar{f}$ closest to $\mathbb{S}_{d-1}$, with ties ignored as occurring with probability zero. The total mass $\mu_\lambda^{f_k}(\mathbb{B}_d)$ coincides with $f_{k;\lambda}$.

Additional basic functionals and measures of interest. Readers only interested in the classic above mentioned functionals of convex hulls can safely skip this paragraph and proceed to the next paragraph describing their scaled versions.

- *Curvature measures.* These objects of integral geometry feature as polynomial coefficients in the *local Steiner formula*. Let $d(x, B)$ stand for the distance between x and B. For a compact convex set $K \subset \mathbb{R}^d$ and a measurable $A \subseteq \partial K$ we have for each $\epsilon \geq 0$

$$\mathrm{Vol}(\{x \in K^c,\ p_K(x) \in A,\ d(x, K) \leq \epsilon\}) = \sum_{k=0}^{d-1} \epsilon^{d-k} \kappa_{d-k} \Phi_k(K; A), \tag{2.6}$$

where $\Phi_k(K; \cdot)$ is the *k-th order curvature measure* of K, see (14.12) and Chapter 14 of [38]. Here $\kappa_j := \pi^{j/2}[\Gamma(1 + j/2)]^{-1}$ is the volume of the j-dimensional unit ball. Usually the curvature measures are defined on the whole of $\mathbb{R}^d$, with support in ∂K. In this paper we shall be interested in the curvature measures $\Phi_k^\lambda(\cdot) := \Phi_k(K_\lambda; \cdot)$ regarded as random measures on $\mathbb{B}_d \supset \partial K$ for formal convenience. In the polytopal case the curvature measures admit a particularly simple form. To see it, for each $x \in \partial K_\lambda$ define the normal cone $N(K_\lambda, x)$ of K_λ at x to be the set of all outward normal vectors to K_λ at x, including the zero vector $\mathbf{0}$, see p. 603 in [38]. Since the normal cone is constant over the relative interiors of faces of K_λ, we can define $N(K_\lambda, f)$ for each $f \in \mathcal{F}_k(K_\lambda),\ k \in \{0, \ldots, d-1\}$, see p. 604 of [38]. Further, for $f \in \mathcal{F}_k(K_\lambda),\ k \in \{0, \ldots, d-1\}$, by the external angle of K_λ at f we understand

$$\gamma(K_\lambda, f) := \frac{1}{\kappa_{d-k}} \mathrm{Vol}_{d-k}(N(K_\lambda, f) \cap \mathbb{B}_d) = \frac{1}{(d-k)\kappa_{d-k}} \sigma_{d-k-1}(N(K_\lambda, f) \cap \mathbb{S}_{d-1}) \tag{2.7}$$

with σ_j *standing for the (non-normalized) j-dimensional spherical surface measure*, see (14.10) in [38]. For each face $f \in \mathcal{F}_k(K_\lambda)$ we write ℓ^f for the k-dimensional volume (Lebesgue-Hausdorff) measure on f regarded as a measure on $\mathbb{B}_d$, that is to say $\ell^f(A) := \mathrm{Vol}_k(A \cap f)$ for measurable $A \subseteq \mathbb{B}_d$. Then, see (14.13) in [38], we have

$$\Phi_k^\lambda(\cdot) = \sum_{f \in \mathcal{F}_k(K_\lambda)} \gamma(K_\lambda, f) \ell^f(\cdot). \tag{2.8}$$

Taking $\Phi_k^\lambda(\mathbb{B}_d)$ yields a representation of the *kth intrinsic volume* $V_k(K_\lambda)$ of the polytope K_λ, namely

$$\Phi_k^\lambda(\mathbb{B}_d) = V_k(K_\lambda) = \sum_{f \in \mathcal{F}_k(K_\lambda)} \gamma(K_\lambda, f)\ell^f(\mathbb{B}_d). \tag{2.9}$$

In this context we make a rather obvious but crucial observation. For a point $u \in \mathbb{S}_{d-1}$ let $M[u; K_\lambda]$ be the set of vertices $x \in f_0(K_\lambda)$ with the property that $h_{K_\lambda}(u) = h_{\{x\}}(u)$. Then, for a face $f \in \mathcal{F}_k(K_\lambda)$ with vertices $x_1, \ldots, x_{k+1}$ we have

$$N(K_\lambda, f) \cap \mathbb{S}_{d-1} = \{u \in \mathbb{S}_{d-1},\ M[u; K_\lambda] = \{x_1, \ldots, x_{k+1}\}\}. \tag{2.10}$$

In geometric terms, the spherical section of the normal cone of K_λ at f coincides with the radial projection of the set of points of $\partial(F(K_\lambda))$ belonging to the boundaries of exactly $k+1$ constituent balls $\partial B_d(x_i/2, |x_i|/2),\ i = 1, \ldots, k+1$. In particular, in view of (2.7), the external angle at f is proportional to the $(d-k-1)$-dimensional surface measure of this projection.

- *Support measures.* These measures generalize the concept of curvature measures and show up in the generalized local Steiner formula. For a compact convex set $K \subset \mathbb{R}^d$ and a measurable $A \subseteq \partial K \times \mathbb{S}_{d-1}$ we have for each $\epsilon \geq 0$

$$\mathrm{Vol}(\{x \in K^c,\ (p_K(x), [x - p_K(x)]/d(x,K)) \in A,\ d(x,K) \leq \epsilon\}) = \sum_{k=0}^{d-1} \epsilon^{d-k} \kappa_{d-k} \Xi_k(K; A) \tag{2.11}$$

where $\Xi_k(K; \cdot)$ is the *k-th order support measure* (or generalized curvature measure) on $\mathbb{R}^d \times \mathbb{S}_{d-1}$, see Theorem 14.2.1 and (14.9) in [38]. To provide a representation of these measures for K_λ, for each $f \in \mathcal{F}_k(K_\lambda),\ k \in \{0, \ldots, d-1\}$, let $\mathcal{U}_{N(K_\lambda,f)\cap\mathbb{S}_{d-1}}$ be the uniform law (normalized $(d-k-1)$-dimensional volume measure) on $N(K_\lambda, f) \cap \mathbb{S}_{d-1}$, that is to say the law of a randomly uniformly chosen direction within the normal cone of K_λ at f. Then we have for $\Xi_k^\lambda(\cdot) := \Xi_k(K_\lambda, \cdot)$

$$\Xi_k^\lambda = \sum_{f \in \mathcal{F}_k(K_\lambda)} \gamma(K_\lambda, f)\ell^f \times \mathcal{U}_{N(K_\lambda,f)\cap\mathbb{S}_{d-1}} \tag{2.12}$$

with $\times$ standing for the usual product of measures; see (14.11) in [38].

- *Projection avoidance functionals.* The representation of the intrinsic volumes of K_λ as the total masses of the corresponding curvature measures, while suitable in the local scaling regime, will turn out to be much less useful in the global scaling regime as leading to asymptotically vanishing add-one cost for related stabilizing functionals, thus precluding normal use of general stabilization theory. To overcome this problem, we shall use the following consequence of the general Crofton's formula, usually going under the name of Kubota's formula, see (5.8) and (6.11) in [38],

$$V_k(K_\lambda) = \frac{d!\kappa_d}{k!\kappa_k(d-k)!\kappa_{d-k}} \int_{G(d,k)} \mathrm{Vol}_k(K_\lambda|L)\nu_k(dL) \tag{2.13}$$

where $G(d,k)$ is the k-th Grassmannian of $\mathbb{R}^d$, ν_k is the normalized Haar measure on $G(d,k)$ and $K_\lambda|L$ is the orthogonal projection of K_λ onto the k-dimensional linear space $L \in G(d,k)$. We shall only focus on the case $k \geq 1$ because for $k = 0$ we have $V_0(K_\lambda) = 1$ by the Gauss-Bonnet theorem; see p. 601 in [38]. Write

$$\int_{G(d,k)} \mathrm{Vol}_k(K_\lambda|L)\nu_k(dL) = \int_{G(d,k)} \int_L [1 - \vartheta_k^{\mathcal{P}_\lambda}(x|L)] d_k(x)\, d\nu_k(L)$$

where $\vartheta_k^{\mathcal{X}}(x|L) := \mathbf{1}_{x \notin \mathrm{conv}(\mathcal{X})|L}$. Putting $x = ru, u \in \mathbb{S}_{d-1},\ r \in [0,1]$, this becomes

$$\int_{G(d,k)} \int_{\mathbb{S}_{d-1}\cap L} \int_0^1 [1 - \vartheta_k^{\mathcal{P}_\lambda}(ru|L)] r^{k-1} dr\, d\sigma_{k-1}(u)\, d\nu_k(L) =$$

$$\int_{G(d,k)} \int_{\mathbb{S}_{d-1}\cap L} \int_0^1 \frac{1}{r^{d-k}} [1 - \vartheta_k^{\mathcal{P}_\lambda}(ru|L)] r^{d-1} dr\, d\sigma_{k-1}(u)\, d\nu_k(L).$$

Noting that $dx = r^{d-1} dr d\sigma_{d-1}(u)$ and interchanging the order of integration we conclude in view of the discussion on p. 590-591 of [38] that the considered expression equals

$$\frac{k\kappa_k}{d\kappa_d} \int_{\mathbb{B}_d} \frac{1}{|x|^{d-k}} \int_{G(\mathrm{lin}[x],k)} [1 - \vartheta_k^{\mathcal{P}_\lambda}(x|L)] d\nu_k^{\mathrm{lin}[x]}(L) dx,$$

where $\mathrm{lin}[x]$ is the 1-dimensional linear space spanned by x, $G(\mathrm{lin}[x],k)$ is the set of k-dimensional linear subspaces of $\mathbb{R}^d$ containing $\mathrm{lin}[x]$, and $\nu_k^{\mathrm{lin}[x]}$ is the corresponding normalized Haar measure, see [38]. Thus, putting

$$\vartheta_k^{\mathcal{X}}(x) := \int_{G(\mathrm{lin}[x],k)} \vartheta_k^{\mathcal{X}}(x|L) d\nu_k^{\mathrm{lin}[x]}(L),\ x \in \mathbb{B}_d, \tag{2.14}$$

and using (2.13) we are led to

$$V_k(\mathbb{B}_d) - V_k(K_\lambda) = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathbb{B}_d} \frac{1}{|x|^{d-k}} \vartheta_k^{\mathcal{P}_\lambda}(x) dx = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathbb{B}_d \setminus K_\lambda} \frac{1}{|x|^{d-k}} \vartheta_k^{\mathcal{P}_\lambda}(x) dx. \tag{2.15}$$

We will refer to $\vartheta_k^{\mathcal{P}_\lambda}$ as the *projection avoidance function* for K_λ.

The large λ asymptotics of the above characteristics of K_λ can be studied in two natural scaling regimes, the *local* and the *global* one, as discussed below.

Local scaling regime and locally re-scaled functionals. The first scaling we consider is referred to as the *local scaling* in the sequel. It stems from the following natural observation. It is known, has been made formal in many ways, and it will also be rigorously discussed in this paper, that if we look at the local behavior of K_λ in the vicinity of two fixed boundary points $u, u' \in \mathbb{S}_{d-1}$, with $\lambda \to \infty$, then these behaviors become asymptotically independent. Even more, if $u' := u'(\lambda)$ approaches u slowly enough as $\lambda \to \infty$, the asymptotic independence is preserved. On the other hand, if the distance between u and $u' := u'(\lambda)$ decays rapidly enough, then both behaviors coincide for large λ and the resulting picture is rather uninteresting. As in [40], it is therefore natural to ask for the frontier of these two asymptotic regimes and to expect that this corresponds to the natural *characteristic scale* between the observation directions u and u' where the crucial features of the local behavior of K_λ are revealed. To determine the right local scaling for our model we begin with the following intuitive argument. To obtain a non-trivial limit behavior we should re-scale K_λ in a neighborhood of $\mathbb{S}_{d-1}$ both in the $d-1$ surfacial (tangential) directions with factor λ^β and radial direction with factor λ^γ with suitable scaling exponents β and γ so that:

- The re-scaling compensates the intensity growth with factor λ as undergone by $\mathcal{P}_\lambda$, that is to say a region in the vicinity of $\mathbb{S}_{d-1}$ with scaling image of a fixed size should host, on average, $\Theta(1)$ points of the re-scaled image of the point process $\mathcal{P}_\lambda$. Since the integral of the intensity density (2.1) scales as $\lambda^{\beta(d-1)}$ standing for the $d-1$ tangential directions, times $\lambda^{\gamma(1+\delta)}$ taking into account the integration over the radial coordinate, we are led to $\lambda^{\beta(d-1)+\gamma(1+\delta)} = \lambda$ and thus
$$\beta(d-1) + \gamma(1+\delta) = 1. \tag{2.16}$$

- The local behavior of the convex hull close to the boundary of $\mathbb{S}_{d-1}$, as described by the locally parabolic structure of s_λ, should preserve parabolic epigraphs, implying for $x \in \mathbb{S}_{d-1}$ that $(\lambda^\beta |x|)^2 = \lambda^\gamma |x|^2$, and thus
$$\gamma = 2\beta. \tag{2.17}$$

Solving the system (2.16,2.17) we end up with
$$\beta = \frac{1}{d+1+2\delta}, \quad \gamma = 2\beta. \tag{2.18}$$

We next describe the scaling transformations for K_λ. To this end, fix a point u_0 on the sphere $\mathbb{S}_{d-1}$ and consider the corresponding *exponential map* $\exp_{u_0} : \mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}_{d-1} \to \mathbb{S}_{d-1}$. Recall that the exponential map $\exp_{u_0}$ maps a vector v of the tangent space T_{u_0} to the point $u \in \mathbb{S}_{d-1}$ such that u lies at the end of the geodesic of length $|v|$ starting at u_0 in the direction of v. Note that $\mathbb{S}_{d-1}$ is geodesically complete in that the exponential map $\exp_{u_0}$ is well defined on the whole tangent space $\mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}_{d-1}$, although it is injective only on $\{v \in T_{u_0}\mathbb{S}_{d-1},\ |v| < \pi\}$. In the sequel we shall write $\exp_{d-1}$ or simply exp and often make the default choice $u_0 := (1, 0, ..., 0)$, unless the explicit choice of u_0 is of importance. Also, we use the isomorphism $T_{u_0}\mathbb{S}_{d-1} \simeq \mathbb{R}^{d-1}$ without further mention and *we shall denote the closure of the injectivity region* $\{v \in T_{u_0}\mathbb{S}_{d-1},\ |v| < \pi\}$ *of the exponential map simply by* $\mathbb{B}_{d-1}(\pi)$. Observe that $\exp(\mathbb{B}_{d-1}(\pi)) = \mathbb{S}_{d-1}$.

Further, consider the following scaling transform T^λ mapping $\mathbb{B}_d$ into $\mathbb{R}^{d-1} \times \mathbb{R}_+$

$$T^\lambda(x) := (\lambda^\beta \exp_{d-1}^{-1}(x/|x|), \lambda^\gamma(1 - |x|)), \tag{2.19}$$

where $\exp^{-1}(\cdot)$ is the inverse exponential map well defined on $\mathbb{S}_{d-1} \setminus \{-u_0\}$ and taking values in the injectivity region $\mathbb{B}_{d-1}(\pi)$. For formal completeness, on the 'missing' point $-u_0$ we let $\exp^{-1}$ admit an arbitrary value, say $(\pi, 0, \ldots, 0)$, and likewise we put $T^\lambda(\mathbf{0}) := (0, \lambda^\gamma)$, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^{d-1}$. Note that T^λ transforms $\mathbb{B}_d \setminus \{\mathbf{0}\}$ onto the solid cylinders

$$\mathcal{R}_\lambda := \lambda^\beta \mathbb{B}_{d-1}(\pi) \times [0, \lambda^\gamma) \tag{2.20}$$

and that T^λ is bijective apart from the afore-mentioned exceptional points.

The transformation T^λ, defined at (2.19), maps the point process $\mathcal{P}_\lambda$ to $\mathcal{P}^{(\lambda)}$, where $\mathcal{P}^{(\lambda)}$ is the dilated Poisson point process in the region $\mathcal{R}_\lambda$ of intensity

$$x = (v, h) \mapsto \left| \nabla^{(1)} \exp_{d-1} \times \ldots \times \nabla^{(d-1)} \exp_{d-1} \right|_{\lambda^{-\beta} v} (1 - \lambda^{-\gamma} h)^{d-1} h^\delta dv dh, \tag{2.21}$$

with $|\nabla^{(1)} \exp_{d-1} \times \ldots \times \nabla^{(d-1)} \exp_{d-1}|_{\lambda^{-\beta} v} d(\lambda^{-\beta} v)$ standing for the spherical surface element at $\exp_{d-1}(\lambda^{-\beta} v)$ and where we have used $\lambda \lambda^{-\beta(d-1)} \lambda^{-\gamma(1+\delta)} = 1$ by (2.16).

In Section 4, following [38], we will embed K_λ, after scaling by T^λ, into a space of parabolic growth processes on $\mathcal{R}_\lambda$. One such process, denoted by $\Psi^{(\lambda)}$ and defined at (4.1), is a *generalized growth process with overlap* whereas the second, a dual process denoted by $\Phi^{(\lambda)}$ and defined in terms of (4.5), is termed the *paraboloid hull process.* Infinite volume counterparts to $\Psi^{(\lambda)}$ and $\Phi^{(\lambda)}$, described fully in Section 3 and denoted by Ψ and Φ, respectively, play a crucial role in describing the asymptotic behavior of our basic functionals and measures of interest, re-scaled as follows:

- The *re-scaled defect support and radius support functions.*

$$\hat{s}_\lambda(v) := \lambda^\gamma s_\lambda(\exp_{d-1}(\lambda^{-\beta} v)),\ v \in \mathbb{R}^{d-1}, \tag{2.22}$$

$$\hat{r}_\lambda(v) := \lambda^\gamma r_\lambda(\exp_{d-1}(\lambda^{-\beta} v)),\ v \in \mathbb{R}^{d-1}. \tag{2.23}$$

- The *re-scaled k-face empirical measure (point process).*

$$\hat{\mu}_\lambda^{f_k} := \sum_{f \in \mathcal{F}_k(K_\lambda)} \delta_{T^\lambda(\mathrm{Top}(f))}. \tag{2.24}$$

- The *re-scaled k-th order curvature measures (2.8) and k-th order support measures (2.12).*

$$\hat{\Phi}_k^\lambda(A) := \lambda^{\beta(d-1)} \sum_{f \in \mathcal{F}_k(K_\lambda)} \gamma(K_\lambda, f) T^\lambda \ell^f(A), \tag{2.25}$$

$$\hat{\Xi}_k^\lambda(A) := \lambda^{\beta(d-1)} \sum_{f \in \mathcal{F}_k(K_\lambda)} \gamma(K_\lambda, f)[T^\lambda \ell^f] \times [T^\lambda \mathcal{U}_{N(K_\lambda,f)\cap \mathbb{S}_{d-1}}](A), \tag{2.26}$$

where $T^\lambda \ell^f(A) := \ell^f[(T^\lambda)^{-1}(A)]$ and A is a Borel subset of $\mathcal{R}_\lambda$.

- The *re-scaled versions of the projection avoidance function (2.14).*

$$\hat{\vartheta}_k^\lambda(x) = \vartheta_k^{\mathcal{P}_\lambda}([T^\lambda]^{-1}(x)),\ x \in \mathcal{R}_\lambda. \tag{2.27}$$

Global scaling regime and globally re-scaled functionals. The asymptotic independence of local convex hull geometries at distinct points of $\mathbb{S}_{d-1}$, as discussed above, suggests that the global behavior of both s_λ and r_λ is, in large λ asymptotics, that of the *white noise*. The same observation holds for the empirical measures $\mu_\lambda^{f_k}$ as well as Φ_k^λ and Ξ_k^λ. Therefore it is natural to consider the corresponding integral characteristics of K_λ and to ask whether, under proper scaling, they converge in law to a Brownian sheet. Define

$$W_\lambda(v) := \int_{\exp([\mathbf{0},v])} s_\lambda(w) d\sigma_{d-1}(w),\ v \in \mathbb{R}^{d-1}, \tag{2.28}$$

and

$$V_\lambda(v) := \int_{\exp([\mathbf{0},v])} r_\lambda(w) d\sigma_{d-1}(w),\ w \in \mathbb{R}^{d-1}, \tag{2.29}$$

where the 'segment' $[\mathbf{0}, v]$ for $v \in \mathbb{R}^{d-1}$ is the rectangular solid in $\mathbb{R}^{d-1}$ with vertices $\mathbf{0}$ and v, that is to say $[\mathbf{0}, v] := \prod_{i=1}^{d-1}[\min(0, v^{(i)}), \max(0, v^{(i)})]$, with $v^{(i)}$ standing for the ith coordinate of v. Although both W_λ and V_λ are defined on the whole of $\mathbb{R}^{d-1}$ for formal convenience, we have $W_\lambda(v) = W_\lambda(w)$ as soon as $[\mathbf{0}, v] \cap \mathbb{B}_d(\pi) = [\mathbf{0}, w] \cap \mathbb{B}_d(\pi)$ and likewise for V_λ. We shall also consider the cumulative values

$$W_\lambda := W_\lambda(\infty) := \int_{\mathbb{S}_{d-1}} s_\lambda(w) d\sigma_{d-1}(w); \quad V_\lambda := V_\lambda(\infty) := \int_{\mathbb{S}_{d-1}} r_\lambda(w) d\sigma_{d-1}(w). \tag{2.30}$$

Note that

$$W_\lambda := W_\lambda(\infty) = \sum_{v,\ v^{(i)} \in \{-\pi, \pi\}} W_\lambda(v) \tag{2.31}$$

and likewise for $V_\lambda(\infty)$. Since the radius-vector function of the Voronoi flower $F(\mathcal{P}_\lambda)$ coincides with the support functional of K_λ, it follows that the volume outside $F(\mathcal{P}_\lambda)$ is asymptotically equivalent to the integral of the defect support functional, which in turn is proportional to the defect mean width. Moreover, in two dimensions the mean width is the ratio of the perimeter to π (see p. 210 of [36]) and so $W_\lambda(\infty)/\pi$ *coincides with* 2 *minus the mean width of* K_λ *and consequently* $W_\lambda(\infty)$ *itself equals* 2π *minus the perimeter of* K_λ *for* $d = 2$. On the other hand, $V_\lambda(\infty)$ is asymptotic to the volume of $\mathbb{B}_d \setminus K_\lambda$, *whence the notation* W *for width and* V *for volume.* To get the desired convergence to a Brownian sheet we put

$$\zeta := \beta(d-1) + 2\gamma = \frac{d+3}{d+1+2\delta} \tag{2.32}$$

and we show that their centered and re-scaled versions

$$\hat{W}_\lambda(v) := \lambda^{\zeta/2}(W_\lambda(v) - \mathbb{E}\, W_\lambda(v)) \text{ and } \quad \hat{V}_\lambda(v) := \lambda^{\zeta/2}(V_\lambda(v) - \mathbb{E}\, V_\lambda(v)) \tag{2.33}$$

converge to a Brownian sheet with an explicit variance coefficient.

Putting the picture together. The remainder of this paper is organized as follows.

Section 3. Though the formulations of our results might suggest otherwise, there are crucial connections between the local and global scaling regimes. These regimes are linked by stabilization and the objective method, the essence of which is to show that the behavior of locally defined processes on the finite volume rectangular solids $\mathcal{R}_\lambda$ defined at (2.20) can be well approximated by the local behavior of a related 'candidate object', either *a generalized growth process* Ψ or a *dual paraboloid hull process* Φ, on an *infinite volume half-space.* While generalized growth processes were developed in [40] in a larger context, our limit theory depends heavily on the dual paraboloid hull process. The purpose of Section 3 is to carefully define these processes and to establish properties relevant to their asymptotic analysis.

Section 4. We show that as $\lambda \to \infty$, both $\hat{s}_\lambda$ and $\hat{r}_\lambda$ converge in law in to continuous path stochastic processes explicitly constructed in terms of the parabolic generalized growth process Ψ and the parabolic fill process Φ, respectively. This adds to Molchanov [24], who considers convergence of (the binomial analog of) $\lambda r_\lambda(u)$ in $\mathbb{S}_{d-1} \times \mathbb{R}_+$, but who does not consider parabolic processes. We shall prove that as $\lambda \to \infty$ the k-face empirical measures $\hat{\mu}_\lambda^{f_k}$ together with the kth order curvature measures $\hat{\Phi}_k^\lambda$ and support measures $\hat{\Xi}_k^\lambda$ converge in law to non-trivial random measures defined in terms of Ψ and Φ.

Section 5. When $d = 2$, after re-scaling in space by a factor of $\lambda^{1/3}$ and in time (height coordinate) by $\lambda^{2/3}$, we use non-asymptotic direct considerations to provide explicit asymptotic expressions for the fidis of $\hat{s}_\lambda$ and $\hat{r}_\lambda$ as $\lambda \to \infty$. These distributions coincide with the fidis of the parabolic growth process Ψ and the parabolic fill process Φ, respectively.

Section 6. Both the parabolic growth process and its dual paraboloid hull process are shown to enjoy a localization property, which expresses in geometric terms a type of spatial mixing. This provides a direct route towards establishing first and second order asymptotics for the convex hull functionals of interest.

Section 7. This section establishes explicit variance asymptotics for the total number of k-faces as well as the intrinsic volumes for the random polytope K_λ. We also establish variance asymptotics and a central limit theorem for the properly scaled integrals of continuous test functions against the empirical measures associated with the functionals under proper scaling.

Section 8. Using the stabilization properties established in Section 6, we shall establish a functional central limit theorem for $\hat{W}_\lambda$ and $\hat{V}_\lambda$, showing that these processes converge as $\lambda \to \infty$ in the space of continuous functions on $\mathbb{R}^{d-1}$ to Brownian sheets with variance coefficients given explicitly in terms of the processes Ψ and Φ, respectively.

Section 9. This section establishes that the distribution functions of the extremal value of s_λ and r_λ both converge to a Gumbel-extreme value distribution as $\lambda \to \infty$.

Section 10. This section uses the results of the previous sections to deduce the limit theory of the typical Poisson-Voronoi and Crofton cells conditioned on having inradius greater than $t > 0$.

Appendix. We derive second order results for the pair correlation function of the point process of extreme points.

3 Paraboloid growth and hull processes

In this section we introduce the paraboloid *growth* and *hull* processes in the upper half-space $\mathbb{R}^{d-1} \times \mathbb{R}_+$ often interpreted as formal *space-time* below, with $\mathbb{R}^{d-1}$ standing for the spatial dimension and $\mathbb{R}_+$ standing for the time – whereas this interpretation is purely formal in the convex hull set-up, it establishes a link to a well established theory of growth processes studied by means of stabilization theory, see below for further details. These processes will turn out to be infinite volume counterparts to finite volume parabolic growth processes, which are defined in the next section, and which are used to describe the behavior of our basic re-scaled functional and measures.

Paraboloid growth processes on half-spaces. We introduce the *paraboloid generalized growth process with overlap* (paraboloid growth process for short), specializing to our present set-up the corresponding general concept defined in Subsection 1.1 of [40] and designed to constitute the asymptotic counterpart of the Voronoi flower $F(K_\lambda)$. Let $\Pi^\uparrow$ be the epigraph of the standard paraboloid $v \mapsto |v|^2/2$, that is

$$\Pi^\uparrow := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+,\ h \geq |v|^2/2\}.$$

Given a locally finite point set $\mathcal{X}$ in $\mathbb{R}^{d-1} \times \mathbb{R}_+$, the paraboloid growth process $\Psi(\mathcal{X})$ is defined as the Boolean model with paraboloid grain $\Pi^\uparrow$ and with germ collection $\mathcal{X}$, namely

$$\Psi(\mathcal{X}) := \mathcal{X} \oplus \Pi^\uparrow = \bigcup_{x \in \mathcal{X}} x \oplus \Pi^\uparrow, \tag{3.1}$$

where $\oplus$ stands for Minkowski addition. The process $\Psi(\mathcal{X})$ arises as the union of upwards paraboloids with apices at the points of $\mathcal{X}$ (see Figure 1), in close analogy to the Voronoi flower $F(\mathcal{X})$ where to each $x \in \mathcal{X}$ we attach a ball $B_d(x/2, |x|/2)$ (which asymptotically scales to an upward paraboloid as we shall see in the sequel) and take the union thereof.

The name *generalized growth process with overlap* comes from the original interpretation of this construction [40], where $\mathbb{R}^{d-1} \times \mathbb{R}_+$ stands for *space-time* with $\mathbb{R}^{d-1}$ corresponding to the *spatial* coordinates and the semi-axis $\mathbb{R}_+$ corresponding to the *time (or height)* coordinate, and where the grain $\Pi^\uparrow$, possibly admitting more general shapes as well, arose as the graph of growth of a germ born at the apex of $\Pi^\uparrow$ and growing thereupon in time with properly varying speed. We say that the process *admits overlaps* because the growth does not stop when two grains overlap, unlike in traditional growth schemes. We shall often use this space-time interpretation and refer to the respective coordinate axes as to the spatial and time (height) axis.

We will be particularly interested in the paraboloid growth process $\Psi := \Psi(\mathcal{P})$, where $\mathcal{P}$ *is a Poisson point process in the upper half-space* $\mathbb{R}^{d-1} \times \mathbb{R}_+$ *with intensity density* $h^\delta dhdv$ at $(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$. The boundary $\partial\Psi$ of the random closed set Ψ constitutes a graph of a continuous function from $\mathbb{R}^{d-1}$ (space) to $\mathbb{R}_+$ (time), also denoted by $\partial\Psi$ in the sequel. In what follows we interpret s_λ as the boundary of a growth process $\Psi^{(\lambda)}$, defined at (4.1) below, on the

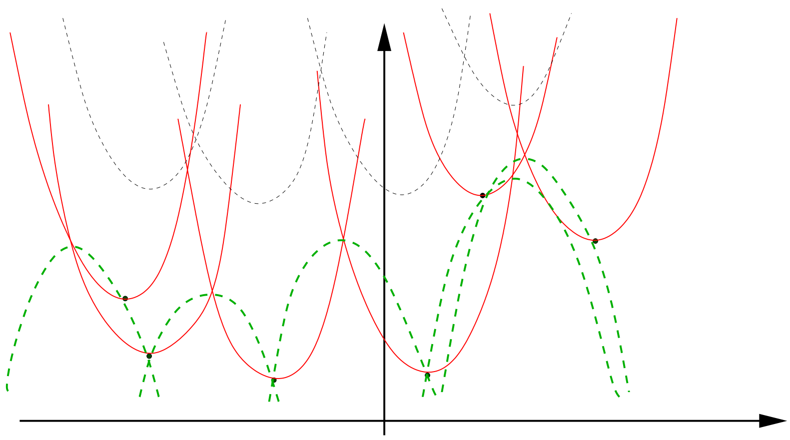

Figure 1: Example of paraboloid and growth processes for $d = 2$

finite region $\mathcal{R}_\lambda$ (recall (2.20)) and we will see that $\partial\Psi$ features as the candidate limiting object for the boundary of $\Psi^{(\lambda)}$.

A germ point $x \in \mathcal{P}$ is called *extreme* in the paraboloid growth process Ψ iff its associated epigraph $x \oplus \Pi^{\uparrow}$ is *not* contained in the union of the paraboloid epigraphs generated by other germ points in $\mathcal{P}$, that is to say

$$x \oplus \Pi^{\uparrow} \not\subseteq \bigcup_{y \in \mathcal{P}, y \neq x} y \oplus \Pi^{\uparrow}. \tag{3.2}$$

Note that to be extreme, it is not necessary that x itself fails to be contained in paraboloid epigraphs of other germs. Write $\mathrm{ext}(\Psi)$ for the set of all extreme points.

Paraboloid hull process on half-spaces. The *paraboloid hull process* Φ can be regarded as the dual to the paraboloid growth process. At the same time, the paraboloid hull process is designed to exhibit geometric properties analogous to those of convex polytopes with *paraboloids* playing the role of *hyperplanes*, with the *spatial coordinates* playing the role of *spherical coordinates* and with the *height/time coordinate* playing the role of the *radial coordinate*. The motivation of this construction is to mimick the convex geometry on second order paraboloid structures in order to describe the local second order geometry of convex polytopes, which dominates their limit behavior in smooth convex bodies. As we will see, this intuition is indeed correct and results in a detailed description of the limit behavior of K_λ.

To proceed with our definitions, we let $\Pi^{\downarrow}$ be the downwards space-time paraboloid hypograph

$$\Pi^{\downarrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R},\ h \leq -|v|^2/2\}. \tag{3.3}$$

The idea behind our interpretation of the paraboloid process is that the shifts of $\Pi^{\downarrow}$ correspond to half-spaces not containing $\mathbf{0}$ in the Euclidean space $\mathbb{R}^d$. Building on this interpretational assumption, we shall argue the *paraboloid convex sets* have properties strongly analogous to those related to the usual concept of convexity. The corresponding proofs are not difficult and will be presented in enough detail to make our presentation self-contained, but it should be emphasized that alternatively the entire argument of this paragraph could be re-written in terms of the following *trick*. Considering the transform $(v,h) \mapsto (v, h+|v|^2/2)$ we see that it maps translates of $\Pi^{\downarrow}$ to half-spaces and thus whenever we make a statement below in terms of paraboloids and claim it is analogous to a standard statement of convex geometry, we can alternatively apply the above auxiliary transform, use the classical result and then transform back to our set-up. We do not choose this option here, finding it more aesthetic to work directly in the paraboloid set-up, but we indicate at this point the availability of this alternative.

To proceed, for any collection $x_1 := (v_1,h_1),\ldots,x_k := (v_k,h_k),\ k \leq d$, of points in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with affinely independent spatial coordinates v_i, we define $\Pi^{\downarrow}[x_1,\ldots,x_k]$ to be the hypograph in $\mathrm{aff}[v_1,\ldots,v_k] \times \mathbb{R}$ of the unique space-time paraboloid in the affine space $\mathrm{aff}[v_1,\ldots,v_k] \times \mathbb{R}$ with quadratic coefficient $-1/2$ and passing through $x_1,\ldots,x_k$. In other words $\Pi^{\downarrow}[x_1,\ldots,x_k]$ is the intersection of $\mathrm{aff}[v_1,\ldots,v_k] \times \mathbb{R}$ and a translate of $\Pi^{\downarrow}$ having all $x_1,\ldots,x_k$ on its boundary; while such translates are non-unique for $k < d$, their intersections with $\mathrm{aff}[v_1,\ldots,v_k]$ all coincide. Recall that the affine hull $\mathrm{aff}[v_1,\ldots,v_k]$ is the set of all affine combinations $\alpha_1 v_1 + \ldots + \alpha_k v_k,\ \alpha_1 + \ldots + \alpha_k = 1,\ \alpha_i \in \mathbb{R}$. Moreover, for $x_1 := (v_1,h_1) \neq x_2 := (v_2,h_2) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, the *parabolic segment* $\Pi^{\square}[x_1,x_2]$ is simply the unique parabolic segment with quadratic coefficient $-1/2$ joining x_1 to x_2 in $\mathrm{aff}[v_1,v_2] \times \mathbb{R}$. More generally, for any collection $x_1 := (v_1,h_1),\ldots,x_k := (v_k,h_k),\ k \leq d$, of points in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with affinely independent spatial coordinates, we define the *paraboloid face* $\Pi^{\square}[x_1,\ldots,x_k]$ by

$$\Pi^{\square}[x_1,\ldots,x_k] := \partial \Pi^{\downarrow}[x_1,\ldots,x_k] \cap [\mathrm{conv}(v_1,\ldots,v_k) \times \mathbb{R}]. \tag{3.4}$$

Clearly, $\Pi^{\square}[x_1,\ldots,x_k]$ is the smallest set containing $x_1,\ldots,x_k$ and with the *paraboloid convexity* property: For any two y_1,y_2 it contains, it also contains $\Pi^{\square}[y_1,y_2]$. In these terms, $\Pi^{\square}[x_1,\ldots,x_k]$

is the *paraboloid convex hull* p-hull($\{x_1, \ldots, x_k\}$). In particular, we readily derive the property

$$\Pi^{\square}[x_1, \ldots, x_i, \ldots, x_k] \cap \Pi^{\square}[x_i, \ldots, x_k, \ldots, x_m] = \Pi^{\square}[x_i, \ldots, x_k], \;\; 1 < i < k. \tag{3.5}$$

Next, we say that $A \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ is *upwards paraboloid convex* (up-convex for short) iff

- for each two $x_1, x_2 \in A$ we have $\Pi^{\square}[x_1, x_2] \subseteq A$,
- and for each $x = (v, h) \in A$ we have $x^{\uparrow} := \{(v, h'),\ h' \geq h\} \subseteq A$.

Whereas the first condition in the definition above is quite intuitive, the second will be seen to correspond to our requirement that $\mathbf{0} \in K_\lambda$ as $\mathbf{0}$ gets transformed to *upper infinity* in the limit of our re-scalings. *By the paraboloid hull (up-hull for short) of $A \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ we shall understand the smallest up-convex set containing A.*

Define the paraboloid hull process Φ as the up-hull of $\mathcal{P}$, that is to say

$$\Phi := \text{up-hull}(\mathcal{P}). \tag{3.6}$$

For $A \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ we put $A^{\uparrow} := \{(v, h'),\ (v, h) \in A \text{ for some } h \leq h'\}$ and observe that if $x_1' \in x_1^{\uparrow},\ x_2' \in x_2^{\uparrow}$, then

$$\Pi^{\square}[x_1', x_2'] \subset [\Pi^{\square}[x_1, x_2]]^{\uparrow} \tag{3.7}$$

and, more generally, by definition of $\Pi^{\square}[x_1, \ldots, x_k]$ and by induction in k, $\Pi^{\square}[x_1', \ldots, x_k'] \subset [\Pi^{\square}[x_1, \ldots, x_k]]^{\uparrow}$. Consequently, we conclude that

$$\Phi = [\text{p-hull}(\mathcal{P})]^{\uparrow}, \tag{3.8}$$

which in terms of our analogy between convex polytopes and paraboloid hulls processes reduces to the trivial statement that a convex polytope containing $\mathbf{0}$ arises as the union of radial segments joining $\mathbf{0}$ to convex combinations of its vertices. This statement is somewhat less uninteresting in the present set-up where $\mathbf{0}$ *disappears* at infinity, and we formulate it here for further use.

Next, we claim that, with probability 1,

$$\Phi = \bigcup_{\{x_1, \ldots, x_d\} \subset \mathcal{P}} [\Pi^{\square}[x_1, \ldots, x_d]]^{\uparrow}, \tag{3.9}$$

which corresponds to the property of d-dimensional polytopes containing $\mathbf{0}$, stating that the convex hull of a collection of points containing $\mathbf{0}$ is the union of all d-dimensional simplices with vertex sets running over all cardinality $(d+1)$ sub-collections of the generating collection which contain $\mathbf{0}$. Subsets $\{x_1, \ldots, x_d\} \subset \mathcal{P}$ have their spatial coordinates affinely independent with probability 1 and thus the right-hand side in (3.9) is a.s. well defined; in the sequel we shall say that points of $\mathcal{P}$ are a.s. *in general position.* Observe that, in view of (3.8) and the fact that $\bigcup_{\{x_1,\ldots,x_d\}\subset\mathcal{P}} \Pi^{\square}[x_1, \ldots, x_d] \subset \text{p-hull}(\mathcal{P})$, (3.9) will follow as soon as we show that

$$\text{p-hull}(\mathcal{P}) \subset \bigcup_{\{x_1,\ldots,x_d\}\subset\mathcal{P}} [\Pi^{\square}[x_1, \ldots, x_d]]^{\uparrow}. \tag{3.10}$$

To establish (3.10) it suffices to show that adding an extra point x_{d+1} in general position to a set $\bar{x} = \{x_1, \ldots, x_d\}$ results in having

$$\text{p-hull}(\bar{x}^+ := \bar{x} \cup \{x_{d+1}\}) \subset \bigcup_{i=1}^{d+1} [\Pi^{\square}[\bar{x}^+ \setminus \{x_i\}]]^{\uparrow}, \tag{3.11}$$

an inductive use of this fact readily yields the required relation (3.10). To verify (3.11) choose $y = (v, h) \in \text{p-hull}(\bar{x}^+)$. Then there exists $y' = (v', h') \in \Pi^{\square}[x_1, \ldots, x_d]$ such that $y \in \Pi^{\square}[y', x_{d+1}]$. Consider the section of $\Pi^{\square}[x_1, \ldots, x_d]$ by the plane $\text{aff}[v', v_{d+1}] \times \mathbb{R}$ and y'' be its point with the lowest height coordinate. Clearly then there exists $x_i,\ i \in \{1, \ldots, d\}$ such that $y'' \in \Pi^{\square}[\bar{x}\setminus\{x_i\}]$. On the other hand, by the choice of y'' and by (3.7), $y \in \Pi^{\square}[y', x_{d+1}] \subset [\Pi^{\square}[y', y'']]^{\uparrow} \cup [\Pi^{\square}[y'', x_{d+1}]]^{\uparrow}$. Consequently, $y \in [\Pi^{\square}[\bar{x}^+ \setminus \{x_i\}]]^{\uparrow} \cup [\Pi^{\square}[\bar{x}]]^{\uparrow}$ which completes the proof of (3.11) and thus also of (3.10) and (3.9).

To formulate our next statement we say that a collection $\{x_1, \ldots, x_d\}$ is *extreme* in $\mathcal{P}$ iff $\Pi^{\square}[x_1, \ldots, x_d] \subset \partial\Phi$. Note that, by (3.7) and (3.9) this is equivalent to having

$$\Phi \cap \Pi^{\downarrow}[x_1, \ldots, x_d] = \Pi^{\square}[x_1, \ldots, x_d]. \tag{3.12}$$

Each such $\Pi^{\square}[x_1, \ldots, x_d]$ is referred to as a *paraboloid sub-face.* Further, say that two extreme collections $\{x_1, \ldots, x_d\}$ and $\{x'_1, \ldots, x'_d\}$ in $\mathcal{P}$ are co-paraboloid iff $\Pi^{\downarrow}[x_1, \ldots, x_d] = \Pi^{\downarrow}[x'_1, \ldots, x'_d]$. By a $(d-1)$-*dimensional paraboloid face* of Φ we shall understand the union of each maximal collection of co-paraboloid sub-faces. Clearly, these correspond to $(d-1)$-dimensional faces of convex polytopes. It is not difficult to check that $(d-1)$-dimensional paraboloid faces of Φ are p-convex and their union is $\partial\Phi$. In fact, since $\mathcal{P}$ is a Poisson process, with probability one all $(d-1)$-dimensional faces of Φ consist of precisely one sub-face; in particular all $(d-1)$-dimensional faces of Φ are bounded. By (3.12) we have for each $(d-1)$-dimensional face f

$$\Phi \cap \Pi^{\downarrow}[f] = f \tag{3.13}$$

which corresponds to the standard fact of the theory of convex polytopes stating that the intersection of a d-dimensional polytope containing $\mathbf{0}$ with a half-space determined by a $(d-1)$-dimensional face and looking away from $\mathbf{0}$ is precisely the face itself. Further, pairs of adjacent $(d-1)$-dimensional paraboloid faces intersect yielding $(d-2)$-dimensional paraboloid manifolds, called $(d-2)$-dimensional paraboloid faces. More generally, $(d-k)$-dimensional paraboloid faces arise as $(d-k)$-dimensional paraboloid manifolds obtained by intersecting suitable k-tuples of adjacent $(d-1)$-dimensional faces. Finally, we end up with zero dimensional faces, which are the *vertices* of Φ and which are easily seen to belong to $\mathcal{P}$. The set of vertices of Φ is denoted by $\mathrm{Vertices}(\Phi)$. In other words, we obtain a full analogy with the geometry of faces of d-dimensional polytopes. To reflect this analogy in our notation, we shall write $\mathcal{F}_k(\Phi)$ for the collection of all k-dimensional faces of Φ. Clearly, $\partial\Phi$ is the graph of a continuous piecewise parabolic function from $\mathbb{R}^{d-1}$ to $\mathbb{R}$. We shall also consider the face empirical measure $\hat{\mu}_\infty^{f_k}$ given, in analogy to (2.5) and (2.24), by

$$\hat{\mu}_\infty^{f_k} := \sum_{f \in \mathcal{F}_k(\Phi)} \delta_{\mathrm{Top}(f)} \tag{3.14}$$

where, as in (2.5), $\mathrm{Top}(f)$ stands for the point of $\overline{f}$ with the smallest height coordinate.

As a consequence of the above description of the geometry of Φ in terms of its faces, particularly (3.13), we conclude that

$$\Phi = \mathrm{cl}\left(\left[\bigcup_{f \in \mathcal{F}_{d-1}(\Phi)} \Pi^{\downarrow}[f]\right]^c\right) = \bigcap_{f \in \mathcal{F}_{d-1}(\Phi)} \mathrm{cl}\left([\Pi^{\downarrow}[f]]^c\right) \tag{3.15}$$

with $\mathrm{cl}(\cdot)$ standing for the topological closure and with $(\cdot)^c$ denoting the complement in $\mathbb{R}^{d-1} \times \mathbb{R}_+$. This is the parabolic counterpart to the standard fact that a convex polytope is the intersection of closed half-spaces determined by its $(d-1)$-dimensional faces and containing $\mathbf{0}$. From (3.15) it follows that for each point $x \notin \Phi$ there exists a translate of $\Pi^{\downarrow}$ containing x but not intersecting Φ hence in particular not intersecting $\mathcal{P}$, which is the paraboloid version of the standard separation lemma of convex geometry. On the other hand, if x is contained in a translate of $\Pi^{\downarrow}$ not hitting $\mathcal{P}$ then $x \notin \Phi$. Consequently

$$\Phi = \left[\bigcup_{x \in \mathbb{R}^{d-1} \times \mathbb{R}_+,\ [x \oplus \Pi^{\downarrow}] \cap \mathcal{P} = \emptyset} x \oplus \Pi^{\downarrow}\right]^c = \bigcap_{x \in \mathbb{R}^{d-1} \times \mathbb{R}_+,\ [x \oplus \Pi^{\downarrow}] \cap \mathcal{P} = \emptyset} \left[x \oplus \Pi^{\downarrow}\right]^c. \tag{3.16}$$

Alternatively, Φ arises as the complement of the morphological opening of $\mathbb{R}^{d-1} \times \mathbb{R}_+ \setminus \mathcal{P}$ with downwards paraboloid structuring element $\Pi^{\downarrow}$, that is to say

$$\Phi^c = [\mathcal{P}^c \ominus \Pi^{\downarrow}] \oplus \Pi^{\downarrow}$$

with $\ominus$ standing for Minkowski erosion. In intuitive terms this means that the complement of Φ is obtained by trying to *fill* $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with downwards paraboloids $\Pi^{\downarrow}$ forbidden to hit any of the Poisson points in $\mathcal{P}$ – the random open set obtained as the union of such paraboloids is precisely Φ^c.

To link the paraboloid hull and growth processes, note that a point $x \in \mathcal{P}$ is a vertex of Φ iff $x \notin \text{up-hull}(\mathcal{P} \setminus \{x\})$. By (3.16) this means that $x \in \text{Vertices}(\Phi)$ iff there exists y such that $[y \oplus \Pi^{\downarrow}] \cap \mathcal{P} = \{x\}$ and, since the set of y such that $y \oplus \Pi^{\downarrow} \ni x$ is simply $x \oplus \Pi^{\uparrow}$, this condition is equivalent to having $x \oplus \Pi^{\uparrow}$ not entirely contained in $[\mathcal{P} \setminus \{x\}] \oplus \Pi^{\uparrow}$. In view of (3.2) means that

$$\text{ext}(\Psi) = \text{Vertices}(\Phi). \tag{3.17}$$

The theory developed in this section admits a particularly simple form when $d = 2$. To see it, say that two points $x, y \in \text{ext}(\Psi)$ are neighbors in Ψ, with notation $x \sim_{\Psi} y$ or simply $x \sim y$, iff there is no point in $\text{ext}(\Psi)$ with its spatial coordinate between those of x and y. Then $\text{Vertices}(\Phi) = \text{ext}(\Psi)$ as in the general case, and $\mathcal{F}_1(\Phi) = \{\Pi^{\square}[x, y],\ x \sim y \in \text{ext}(\Psi)\}$. In this context it is also particularly easy to display the relationships between the parabolic growth process Ψ and the parabolic hull process Φ in terms of the analogous relations between the convex hull K_λ and the Voronoi flower $F(\mathcal{P}_\lambda)$ upon the transformation (2.19) in large λ asymptotics. To this end, see Figure 2 and note that in large λ asymptotics we have

- The extreme points in Ψ, coinciding with $\text{Vertices}(\Phi)$, correspond to the vertices of K_λ.
- Two points $x, y \in \text{ext}(\Psi)$ are neighbors $x \sim y$ iff the corresponding vertices of K_λ are adjacent, that is to say connected by an edge of ∂K_λ.
- The circles $\mathbb{S}_1(x/2, |x|/2)$ and $\mathbb{S}_1(y/2, |y|/2)$ of two adjacent vertices x, y of K_λ, whose pieces mark the boundary of the Voronoi flower $F(\mathcal{P}_\lambda)$, are easily seen to have their unique non-zero intersection point z collinear with x and y. Moreover, z minimizes the distance to 0 among the points on the $\overline{xy}$-line and $\overline{xy} \perp \overline{0z}$. For the parabolic processes this is reflected by the fact that the intersection point of two upwards parabolae with apices at two neighboring points x and y of $\text{Vertices}(\Phi) = \text{ext}(\Psi)$ coincides with the apex of the downwards parabola $\Pi^{\downarrow}[x, y]$ as readily verified by a direct calculation.

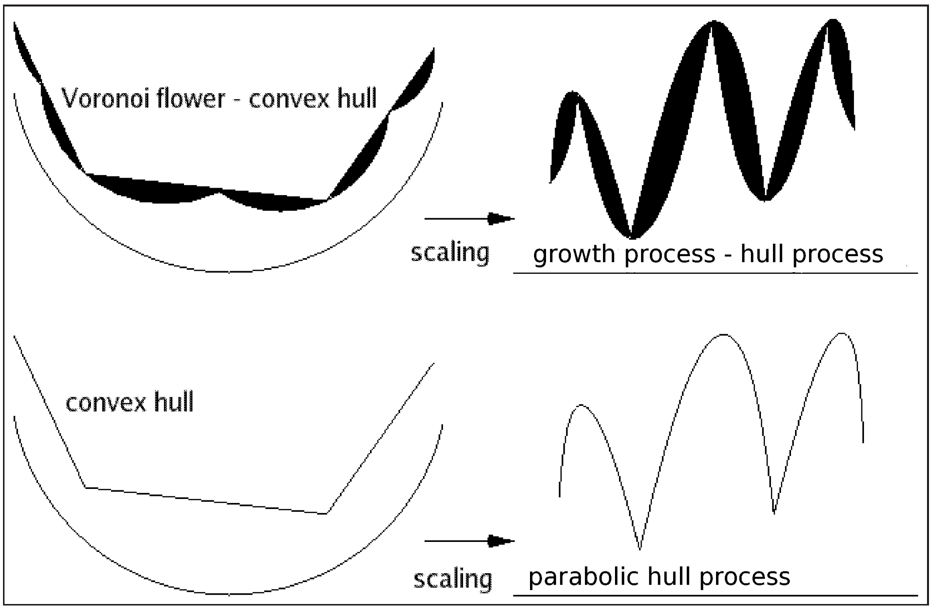

Figure 2: Convex hull, Voronoi flower, and their scaling limits

- Finally, the relation (3.15) becomes here $\Phi = \bigcap_{x\sim y\in \mathrm{ext}(\Psi)} \mathrm{cl}\left([\Pi^{\downarrow}[x,y]]^c\right)$ which is reflected by the fact that K_λ coincides with the intersection of all closed half-spaces containing $\mathbf{0}$ determined by segments of the convex hull boundary ∂K_λ.

Paraboloid curvature and support measures (Infinite volume). The analogy between the paraboloid hull process Φ and the local geometry of convex polytopes extends further. In particular, we can and will define in a natural way the *paraboloid curvature measures* paralleling the construction (2.8) and characterizing the limit behavior of Φ_k^λ.

An interesting feature of paraboloid curvature measures is that their construction involves simultaneously the paraboloid growth and hull processes. To proceed, for a point $v \in \mathbb{R}^{d-1}$ let $M[v;\Psi]$ be the set of points $x = (v_x, h_x) \in \mathrm{ext}(\Psi) = \mathrm{Vertices}(\Phi)$ with the property that $\partial\Psi(v) = h_x + |v - v_x|^2$, that is to say the boundary point of Ψ with spatial coordinate v lies on the boundary of the paraboloid $x \oplus \Pi^{\uparrow}$. Then, for a face $f \in \mathcal{F}_k(\Phi)$ with vertices $x_1, \ldots, x_{k+1} \in \mathrm{ext}(\Psi) = \mathrm{Vertices}(\Phi)$ we define the *paraboloid normal cone* $N(\Phi, f)$ of Φ at f in analogy to (2.10),

$$N(\Phi, f) := \{v \in \mathbb{R}^{d-1},\ M[v;\Phi] = \{x_1, \ldots, x_{k+1}\}\} \times \mathbb{R}_+. \qquad (3.18)$$

In other words, the normal cone of Φ at f is the union of vertical rays in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ determined by the boundary points of Ψ lying on the boundaries of precisely k constituent paraboloids $\partial[x_i \oplus$

$\Pi^{\uparrow}]$, $i = 1, \ldots, k$. Note that the normal cones of faces of $f \in \mathcal{F}_k(\Phi), k \in \{0, 1, ..., d-1\}$, form a random partition of $\mathbb{R}^{d-1}$. Next, we define the *paraboloid external angle* $\gamma(\Phi, f)$ of Φ at f in analogy to (2.7)

$$\gamma(\Phi, f) := \frac{1}{(d-k)\kappa_{d-k}} \mathrm{Vol}_{d-k-1}(N(\Phi, f) \cap \mathbb{R}^{d-1}). \tag{3.19}$$

Finally, for a paraboloid face $f \in \mathcal{F}_k(\Phi)$ we write ℓ_*^f for the measure on $\mathbb{R}^{d-1} \times \mathbb{R}_+$ given for a measurable A by $\ell_*^f(A) := \mathrm{Vol}_k(\pi_{\mathbb{R}^{d-1}}(A \cap f))$, where for $x := (v, h)$ we let $\pi_{\mathbb{R}^{d-1}}(x) = v$ be its spatial projection. Having introduced all necessary ingredients, we now define the (infinite volume) *paraboloid curvature measures* $\Theta_k^\infty := \Theta_k$ of Φ by

$$\Theta_k^\infty(\cdot) := \Theta_k(\cdot) := \sum_{f \in \mathcal{F}_k(\Phi)} \gamma(\Phi, f) \ell_*^f(\cdot), \tag{3.20}$$

in analogy to (2.8). Likewise, we construct the *paraboloid support measures*

$$\Lambda_k^\infty(\cdot) := \Lambda_k(\cdot) := \sum_{f \in \mathcal{F}_k(\Phi)} \gamma(\Phi, f) \ell_*^f \times \mathcal{U}_{N(\Phi, f) \cap \mathbb{R}^{d-1}}(\cdot), \tag{3.21}$$

where $\mathcal{U}_{N(\Phi, f) \cap \mathbb{R}^{d-1}}$ is, as usual, the uniform law (normalized $(d-k-1)$-dimensional volume) on $N(\Phi, f) \cap \mathbb{R}^{d-1}$. Rather than use the full notation Θ_k^∞ and Λ_k^∞, we simplify it to Θ_k and Λ_k. We conclude this paragraph by defining the paraboloid *avoidance function* $\hat{\vartheta}_k^\infty$, $k \geq 1$. To this end, for each $x := (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ let $x^{\updownarrow} := \{(v, h'),\ h' \in \mathbb{R}\}$ be the infinite vertical ray (line) determined by x and let $A(x^{\updownarrow}, k)$ be the collection of all k-dimensional affine spaces in $\mathbb{R}^d$ containing $x^{\updownarrow}$, regarded as the asymptotic equivalent of the restricted Grassmannian $G(\mathrm{lin}[x], k)$ considered in the definition (2.14) of the non-rescaled function ϑ_k^λ. Next, for $L \in A(x^{\updownarrow}, k)$ we define the orthogonal paraboloid surface $\Pi^\perp[x; L]$ to L at x given by

$$\Pi^\perp[x; L] := \{x' = (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R},\ (x - x') \perp L,\ h' = h - d(x, x')^2/2\}. \tag{3.22}$$

Note that this is an analog of the usual orthogonal affine space $L^\perp + x$ to L at x, with the second order parabolic correction typical in our asymptotic setting – recall that non-radial hyperplanes get asymptotically transformed onto downwards paraboloids. Further, for $L \in A(x^{\updownarrow}, k)$ we put

$$\vartheta_k^\infty(x|L) := \mathbf{1}_{\Pi^\perp[x;L] \cap \Phi = \emptyset}.$$

Observe that this is a direct analog of $\vartheta_k^\lambda(x|L)$ assuming the value 1 precisely when $x \notin K_\lambda|L \Leftrightarrow [L^\perp + x] \cap K_\lambda = \emptyset$. Finally, in full analogy to (2.14) set

$$\vartheta_k^\infty(x) = \int_{A(x^{\updownarrow}, k)} \vartheta_k^\infty(x|L) \mu_k^{x^{\updownarrow}}(dL) \tag{3.23}$$

with $\mu_k^{x^{\updownarrow}}$ standing for the normalized Haar measure on $A(x^{\updownarrow}, k)$; see p. 591 in [38].

Duality relations between growth and hull processes. As already signaled, there are close relationships between the paraboloid growth and hull processes, which we refer to as *duality.* Here we discuss these connections in more detail. The first observation is that

$$\Psi = \Phi \oplus \Pi^{\uparrow} = \mathrm{Vertices}(\Phi) \oplus \Pi^{\uparrow}. \tag{3.24}$$

This is verified either directly by the construction of Φ and Ψ, or, less directly but more instructively, by using the fact, established in detail in Section 4 below, that Φ arises as the scaling limit of K_λ whereas Ψ is the scaling limit of the Voronoi flower

$$F(\mathcal{P}_\lambda) = \bigcup_{x \in \mathcal{P}_\lambda} B_d(x/2, |x|/2) = \bigcup_{x \in \mathrm{Vertices}(K_\lambda)} B_d(x/2, |x|/2),$$

defined at (2.3) and then by noting that the balls $B_d(x/2, |x|/2)$ asymptotically either scale into upward paraboloids or they 'disappear at infinity'; see the proof of Theorem 4.1 below and recall that the support function of K_λ coincides with the radius-vector function of $F(K_\lambda)$ as soon as $\mathbf{0} \in K_\lambda$ (which, recall, happens with overwhelming probability). Thus, it is straightforward to transform Φ into Ψ. To construct the dual transform, say that $v \in \mathbb{R}^{d-1}$ is an *extreme direction* for Ψ if $\partial\Psi$ admits a local maximum at v. Further, say that $x \in \partial\Psi$ is an extreme directional point for Ψ, written $x \in \text{ext-dir}(\Psi)$, iff $x = (v, \partial\Psi(v))$ for some extreme direction v. Then we have

$$\Phi^c = \Psi^c \oplus \Pi^{\downarrow} \ \text{ and } \ \mathrm{cl}(\Phi^c) = \text{ext-dir}(\Psi) \oplus \Pi^{\downarrow}. \tag{3.25}$$

Again, this can be directly proved, yet it is more appealing to observe that this statement is simply an asymptotic counterpart of the usual procedure of restoring the convex polytope K_λ given its support function. Indeed, the complement of the polytope arises as the union of all half-spaces of the form $H_x := \{y \in \mathbb{R}^d,\ \langle y - x, x\rangle \geq 0\}$ (asymptotically transformed onto suitable translates of $\Pi^{\downarrow}$ under the action of $T^\lambda,\ \lambda \to \infty$) with x ranging through $x = ru,\ r > h_{K_\lambda}(u),\ r \in \mathbb{R},\ u \in \mathbb{S}_{d-1}$ which corresponds to taking x in the epigraph of h_{K_λ} (transformed onto Ψ^c in our asymptotics). This explains the first equality in (3.25). The second one comes from the fact that it is enough in the above procedure to consider half-spaces H_x for x in extreme directions only, corresponding to directions orthogonal to $(d-1)$-dimensional faces of K_λ and marked by local minima of the support function h_{K_λ} (asymptotically mapped onto local maxima of $\partial\Psi$). It is worth noting that all extreme directional points of Ψ arise as d-fold intersections of boundaries of upwards paraboloids $\partial[x \oplus \Pi^{\uparrow}],\ x \in \mathrm{ext}(\Psi)$, although not all such intersections give rise to extreme directional points (they do so precisely when the apices of d intersecting upwards paraboloids are vertices of the same $(d-1)$-dimensional face of Φ, which is not difficult to prove but which is not needed here).

4 Local scaling limits

Having introduced the paraboloid growth and hull processes Ψ and Φ, respectively, we now establish local scaling results for the local processes $\hat{s}_\lambda$ and $\hat{r}_\lambda$, the empirical measures $\hat{\mu}_\lambda^{f_k}$, as well as the curvature and support measures given respectively by Φ_k^λ and Ξ_k^λ. Our first result shows that the boundaries of Ψ and Φ are the scaling limit of the graphs of $\hat{s}_\lambda$ and $\hat{r}_\lambda$, respectively. This adds to Molchanov [24], who establishes convergence of $nr(u, \{X_i\}_{i=1}^n)$, where X_i are i.i.d. uniform in the unit ball. It also adds to Eddy [16], who considers convergence of the properly scaled defect support function for i.i.d. random variables with a circularly symmetric standard Gaussian distribution. Recall that $B_d(x, r)$ stands for the d-dimensional radius r ball centered at x.

Theorem 4.1 *For any $R > 0$, with $\lambda \to \infty$ the random functions $\hat{s}_\lambda$ and $\hat{r}_\lambda$ converge in law respectively to $\partial\Psi$ and $\partial\Phi$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ of continuous functions on $B_{d-1}(\mathbf{0}, R)$ endowed with the supremum norm.*

Proof. The convergence in law for $\hat{s}_\lambda$ may be shown to follow from the more general theory of generalized growth processes developed in [40], but we provide here a short argument specialized to our present set-up. Recall that we place ourselves on the event that $\mathbf{0} \in K_\lambda$ which is exponentially unlikely to fail as $\lambda \to \infty$ and thus, for our purposes, may be assumed to hold without loss of generality. Further, we note that the support function $h_{\{x\}}(u)$ of a single point $x \in \mathbb{B}_d$ is given by

$$h_{\{x\}}(u) = |x| \cos(d_{\mathbb{S}_{d-1}}(u, x/|x|))$$

with $d_{\mathbb{S}_{d-1}}$ standing for the geodesic distance in $\mathbb{S}_{d-1}$.

Recalling the definition of $\mathcal{P}^{(\lambda)}$ at (2.21) and writing $x := (v_x, h_x)$ for the points in $\mathcal{P}^{(\lambda)}$ shows that under the transformation T^λ, we can write $\hat{s}_\lambda(v),\ v \in \lambda^\beta \mathbb{B}_{d-1}(\pi)$, as

$$\begin{aligned}
\hat{s}_\lambda(v) &= \lambda^\gamma \left(1 - \max_{x=(v_x,h_x)\in\mathcal{P}^{(\lambda)}} [1 - \lambda^{-\gamma} h_x][\cos[\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v_x))]]\right) \\
&= \lambda^\gamma \min_{x\in\mathcal{P}^{(\lambda)}} \left[1 - (1 - \lambda^{-\gamma} h_x)(1 - (1 - \cos[\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v_x))]))\right] \\
&= \min_{x\in\mathcal{P}^{(\lambda)}} \left[h_x + \lambda^\gamma(1 - \cos(\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v_x)))) - \right. \\
&\qquad \left. h_x(1 - \cos[\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v_x))])\right].
\end{aligned}$$

Thus, by (2.2) and (2.22), the graph of $\hat{s}_\lambda$ coincides with the lower boundary of the following *generalized growth process*

$$\Psi^{(\lambda)} := \bigcup_{x \in \mathcal{P}^{(\lambda)}} [\Pi^\uparrow]_x^{(\lambda)} \tag{4.1}$$

where for $x := (v_x, h_x)$ we have

$$[\Pi^\uparrow]_x^{(\lambda)} := \{(v,h),\ h \geq h_x + \lambda^\gamma(1 - \cos[\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v_x))]) -$$

$$h_x(1 - \cos[\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v_x))]\}. \tag{4.2}$$

We now claim that the lower boundary of the process $\Psi^{(\lambda)}$ converges in law in $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ to $\partial\Psi$. Indeed this follows readily by:

- Using Lemma 3.2 in [40] to conclude that, uniformly in λ large enough,

  $$P[\sup_{v \in B_{d-1}(\mathbf{0},R)} \partial\Psi^{(\lambda)}(\tau) \geq H] \leq C[R] \exp(-c[H^{(d+1)/2} \wedge R^{d-1} H^{1+\delta}]) \tag{4.3}$$

  with $c > 0$ and $C[R] < \infty$ (note that the extra term $R^{d-1}H^{1+\delta}$ in the exponent corresponds to the probability of having $B_{d-1}(\mathbf{0}, R) \times [0, H]$ devoid of points of $\mathcal{P}$ and $\mathcal{P}^{(\lambda)}$); that is to say for H large enough, with overwhelming probability, over the spatial region $B_{d-1}(\mathbf{0}, R)$ the lower boundaries of $\Psi^{(\lambda)}$ and Ψ do not reach heights exceeding H.

- Noting that, by (2.21), upon restriction to $B_{d-1}(\mathbf{0}, R) \times [0, H]$, the point process $\mathcal{P}^{(\lambda)}$ converges in variational distance to the corresponding restriction of $\mathcal{P}$, see Theorem 3.2.2 in [32]. Thus, $\mathcal{P}$ and $\mathcal{P}^{(\lambda)}$ can be coupled on a common probability space so that with overwhelming probability $\mathcal{P} \cap [B_{d-1}(\mathbf{0}, R) \times [0, H]] = \mathcal{P}^{(\lambda)} \cap [B_{d-1}(\mathbf{0}, R) \times [0, H]]$. This fact is referred to as the 'total variation convergence on compacts' in the sequel.

- Observing that for each R, H there exist R' and H' such that for all λ large enough the behavior of $\Psi^{(\lambda)}$ and Ψ restricted to $B_{d-1}(\mathbf{0}, R) \times [0, H]$ only depends on the restriction to $B_{d-1}(\mathbf{0}, R') \times [0, H']$ of the process $\mathcal{P}^{(\lambda)}$ and $\mathcal{P}$ respectively. For instance in the case of Ψ it is enough that the region $B_{d-1}(\mathbf{0}, R') \times [0, H']$ contain the apices of all translates of $\Pi^\uparrow$ which hit $B_{d-1}(\mathbf{0}, R) \times [0, H]$.

- Taylor-expanding the cosine function up to second order, recalling $\gamma = 2\beta$ from (2.17), and noting that, upon restriction to $B_{d-1}(\mathbf{0}, R) \times [0, H]$, in large λ asymptotics, the graph of the lower boundary of $[\Pi^\uparrow]_x^{(\lambda)},\ x \in \mathcal{P}^{(\lambda)}$, gets eventually within arbitrarily small sup norm distance ϵ of the graph of the lower boundary of the paraboloid $v \mapsto h_x + |v - v_x|^2/2$ that is to say $x \oplus \partial\Pi^{\uparrow}$.

- Finally, putting the above observations together and recalling the definition (3.1) of the paraboloid growth process and relations (2.2,2.22).

This shows Theorem 4.1 for $\hat{s}_\lambda$.

To proceed with the case of $\hat{r}_\lambda$ observe that under the transformation T^λ, the spherical cap

$$\mathrm{cap}_\lambda[v^*, h^*] := \{y \in \mathbb{B}_d,\ \langle y, \exp_{d-1}(\lambda^{-\beta} v^*)\rangle \geq 1 - \lambda^{-\gamma} h^*\},\quad (v^*, h^*) \in \mathbb{R}^{d-1} \times \mathbb{R}^+, \tag{4.4}$$

transforms into

$$\mathrm{cap}^{(\lambda)}[v^*, h^*] := \{(v, h),\ h \leq \lambda^\gamma \max(0, 1 - (1 - \lambda^{-\gamma} h^*)/\cos(\lambda^{-\beta} d_{\mathbb{S}_{d-1}}(\exp_{d-1}(v), \exp_{d-1}(v^*))))\}.$$

As in the case of $\hat{s}_\lambda$ above, the process $\mathcal{P}_\lambda$ transforms into $\mathcal{P}^{(\lambda)}$. Consequently, using the fact that $\mathbb{B}_d \setminus K_\lambda$ is the union of all spherical caps not hitting any of the points in $\mathcal{P}_\lambda$, we conclude that under the mapping T^λ the complement of K_λ in $\mathbb{B}_d$ gets transformed into the union

$$\bigcup \{\mathrm{cap}^{(\lambda)}[v^*, h^*];\ (v^*, h^*) \in \mathcal{R}_\lambda,\ \mathrm{cap}^{(\lambda)}[v^*, h^*] \cap \mathcal{P}^{(\lambda)} = \emptyset\}. \tag{4.5}$$

Denote by $\Phi^{(\lambda)}$ the complement of this union in $\mathbb{R}^{d-1} \times \mathbb{R}_+$. Our further argument parallels that for $\hat{s}_\lambda$ above:

- We apply again Lemma 3.2 in [40], quoted as (4.3) here, to get rid, with overwhelming probability, of vertices in $\Psi^{(\lambda)}$ and Ψ of very high height coordinates.
- We recall that $\mathcal{P}^{(\lambda)}$ converges to $\mathcal{P}$ in total variation on compacts as $\lambda \to \infty$.
- We note that both Φ and $\Phi^{(\lambda)}$ are locally determined in the sense that for any $R, H, \epsilon > 0$ there exist $R', H' > 0$ with the property that, with probability at least $1 - \epsilon$, the restrictions of Φ and $\Phi^{(\lambda)}$ to $B_{d-1}(\mathbf{0}, R) \times [0, H]$ are determined by the restrictions to $B_{d-1}(\mathbf{0}, R') \times [0, H']$ of $\mathcal{P}^{(\lambda)}$ and $\mathcal{P}$ respectively. This because for the geometry of Φ within $B_{d-1}(\mathbf{0}, R) \times [0, H]$ to be affected by the status of a point $y \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, there should exist a translate of $\Pi^{\downarrow}$
  - hitting $B_{d-1}(\mathbf{0}, R) \times [0, H]$,
  - and containing x on its boundary,
  - and devoid of other points of $\mathcal{P}$,

whence the probability of such influence being exerted by a point far away tends to 0 with the distance of y from $B_{d-1}(\mathbf{0}, R) \times [0, H]$. The argument for $\Phi^{(\lambda)}$ and $\mathcal{P}^{(\lambda)}$ is analogous. Statements of this kind, going under the general name of stabilization, will be discussed in more detail in Section 6.

- We Taylor-expand the cosine function up to second order to see that the upper boundary of $\mathrm{cap}^{(\lambda)}[v^*, h^*]$ is uniformly approximated on compacts by that of $(v^*, h^*) \oplus \Pi^{\downarrow}$ in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ as $\lambda \to \infty$, that is to say the corresponding functions converge in supremum norm on compacts.

- We combine the above observations, recall the relations (2.4, 2.23) and use (4.5) to conclude that $\hat{r}_\lambda$ converges in law in supremum norm on compacts to the continuous function determined by the upper boundary of the process

$$\bigcup_{y \in \mathbb{R}^{d-1} \times \mathbb{R}_+,\ [y \oplus \Pi^{\downarrow}] \cap \mathcal{P} = \emptyset} y \oplus \Pi^{\downarrow}$$

which coincides with $\partial \Phi$ in view of (3.16).

The proof of Theorem 4.1 is hence complete. $\square$

Our next statement provides a local limit description of the k-face empirical measures $\hat{\mu}_\lambda^{f_k}$ defined at (2.24). Recall that the vague topology on the space of measures is the weakest topology which makes continuous the integration of compactly supported continuous functions. This definition carries over to the case of point configurations interpreted as counting measures.

Theorem 4.2 *For each $k \in \{0, 1, \ldots, d-1\}$ the point process $\hat{\mu}_\lambda^{f_k}$ on $\mathbb{R}^{d-1} \times \mathbb{R}_+$ converges in law in the vague topology to $\hat{\mu}_\infty^{f_k}$ as $\lambda \to \infty$.*

Proof. The random measure $\hat{\mu}_\lambda^{f_k}$ arises as a deterministic function, say $h_\lambda(\cdot)$, of the processes $\mathcal{P}^{(\lambda)}$ and $\Phi^{(\lambda)}$ and, likewise $\hat{\mu}_\infty^{f_k}$ defined at (3.14) is a function, say $h(\cdot)$, of processes $\mathcal{P}$ and Φ. In fact, there is some redundancy in this statement because $\Phi^{(\lambda)}$ is determined by $\mathcal{P}^{(\lambda)}$ and so is Φ by $\mathcal{P}$, yet our claim is valid and its form is convenient for our further argument. Next, consider the discrete topology $\mathcal{T}_1$ on compacts for locally finite point sets $\mathcal{X}$ (i.e., take an increasing sequence $C_m,\ m = 1, 2, ...$ of compacts, $C_m \uparrow \mathbb{R}^{d-1} \times \mathbb{R}_+$, and let the distance between locally finite $\mathcal{X}_1$ and $\mathcal{X}_2$ be $\sum_{m=1}^{\infty} 2^{-m} \mathbf{1}_{\mathcal{X}_1 \cap C_m \neq \mathcal{X}_2 \cap C_m}$) and consider the topology $\mathcal{T}_2$ of uniform functional convergence on compacts for the processes $\Phi^{(\lambda)}$ and Φ (identified with the functions graphed by their lower boundaries), and let $\mathcal{T} := \mathcal{T}_1 \times \mathcal{T}_2$ be the resulting product topology. From the proof of Theorem 4.1

it can be seen that on converging sequences of arguments $\omega_\lambda \to \omega$ with ω_λ standing for $(\mathcal{P}^{(\lambda)}, \Phi^{(\lambda)})$ and ω for $(\mathcal{P}, \Phi)$, we have $h_\lambda(\omega_\lambda) \to h(\omega)$ in the topology $\mathcal{T}$. This allows us to use a version of the standard continuous mapping theorem for convergence in law, stated as Theorem 5.5 in [9] to conclude the proof of Theorem 4.2. $\square$

Further, we characterize local scaling asymptotics of the curvature measures $\hat{\Phi}_k^\lambda$ at (2.25).

Theorem 4.3 *For each $k \in \{0, 1, \ldots, d-1\}$ the random measures $\hat{\Phi}_k^\lambda$ on $\mathbb{R}^{d-1} \times \mathbb{R}_+$ converge in law in the vague topology to Θ_k as $\lambda \to \infty$.*

Proof. We proceed as in the proof of Theorem 4.2. First, we note that $\hat{\Phi}_k^\lambda$ arises as a deterministic function of $(\mathcal{P}^{(\lambda)}, \Psi^{(\lambda)}, \Phi^{(\lambda)})$ as specified by (2.7,2.8,2.10) and (2.25). Likewise, Θ_k is a deterministic function of $(\mathcal{P}, \Psi, \Phi)$ as follows by (3.18,3.19) and (3.20). Thus, in order to apply the continuous mapping Theorem 5.5 in [9] it is enough to show that whenever $(\mathcal{P}, \Psi, \Phi)$ is asymptotically approximated by $(\mathcal{P}^{(\lambda)}, \Psi^{(\lambda)}, \Phi^{(\lambda)})$ then so is Θ_k by $\hat{\Phi}_k^\lambda$. To this end, we place ourselves in context of the proof of Theorem 4.1 and observe first that whenever a face $f \in \mathcal{F}_k(\Phi)$ is approximated by $T^\lambda(f^\lambda)$, $f^\lambda \in \mathcal{F}_k(K_\lambda)$ as $\lambda \to \infty$ then $T^\lambda \ell^{f^\lambda}$ approximates $\lambda^{-\beta k} \ell_*^f$. Note that the prefactor $\lambda^{-\beta k}$ is due to the λ^β-re-scaling of spatial dimensions of the k-dimensional face f^λ. There is no asymptotic prefactor corresponding to the height dimension because the order of height fluctuations for $f^\lambda \approx (T^\lambda)^{-1}(f)$ is $O(\lambda^{-\gamma}) = O(\lambda^{-2\beta})$ which is negligible compared to the order $O(\lambda^{-\beta})$ of spatial size of f^λ. Further, with $(\mathcal{P}, \Psi, \Phi)$ approximated by $(\mathcal{P}^{(\lambda)}, \Psi^{(\lambda)}, \Phi^{(\lambda)})$ we have $\gamma(\Phi, f)$ approximated by $\lambda^{-\beta(d-k-1)} \gamma(K_\lambda, f^\lambda)$ because $\gamma(K_\lambda, f^\lambda)$ arises as $(d-k-1)$-dimensional volume undergoing spatial scaling with prefactor $\lambda^{-\beta(d-k-1)}$, see (2.7) and (3.19). Consequently, using (2.8) and (3.20) we see that Θ_k is approximated in the sense of vague topology by $\lambda^{\beta(d-1)} \sum_{f \in \mathcal{F}_k(K_\lambda)} \gamma(K_\lambda, f) T^\lambda \ell^f$ which is precisely $\hat{\Phi}_k^\lambda$. The proof of Theorem 4.3 is hence complete. $\square$

Finally, we give an asymptotic description of the support measures $\hat{\Xi}_k^\lambda$ defined at (2.26).

Theorem 4.4 *For each $k \in \{0, 1, \ldots, d-1\}$ the random measures $\hat{\Xi}_k^\lambda$ on $\mathbb{R}^{d-1} \times \mathbb{R}_+$ converge in law in the vague topology to Λ_k as $\lambda \to \infty$.*

The proof of this result is fully analogous to that of Theorem 4.3 and is therefore omitted.

5 Exact distributional results for scaling limits

This section is restricted to dimension $d = 2$ and to the homogeneous Poisson point process in the unit-disk. Here we provide explicit formulae for the fidis of the processes $\hat{s}_\lambda$ and $\hat{r}_\lambda$ and give their explicit asymptotics, confirming a posteriori the existence of the limiting parabolic growth and hull processes of Section 3. Second-order results for the point process of extremal points are deduced in the appendix.

5.1 The process $\hat{s}_\lambda$

This subsection calculates the distribution of $s(\theta_0, \mathcal{P}_\lambda)$ and establishes the convergence of the fidis of both the process and its re-scaled version. Taking advantage of the two-dimensional set-up, throughout this section we identify the unit sphere $\mathbb{S}_1$ with the segment $[0, 2\pi)$, whence the notation $s(\theta, \cdot)$, $\theta \in [0, 2\pi)$, and likewise for the radius-vector function $r(\theta, \cdot)$. A first elementary result is the following:

Lemma 5.1 *For every $h > 0$, $v_0 \in [0, 2\pi)$, and $\lambda > 0$, we have*

$$P[s(v_0, \mathcal{P}_\lambda) \geq h] = \exp\{-\lambda(\arccos(1-h) - \sqrt{2h - h^2}(1-h))\}.$$

Proof. Notice that $(s(v_0, \mathcal{P}_\lambda) \geq h)$ is equivalent to $\mathrm{cap}_1[v_0, h] \cap \mathcal{P}_\lambda = \emptyset$, where $\mathrm{cap}_1[v_0, h]$ is defined at (4.4). Since the Lebesgue measure $\ell(\mathrm{cap}_1[v_0, h])$ of $\mathrm{cap}_1[v_0, h]$ satisfies

$$\ell(\mathrm{cap}_1[v_0, h]) = \arccos(1-h) - (1-h)\sqrt{2h - h^2}, \tag{5.1}$$

the lemma follows by the Poisson property of the process $\mathcal{P}_\lambda$. $\square$

We focus on the asymptotic behaviour of the process s when λ is large. When we scale in space, we obtain the fidis of white noise and when we scale in both time and space to get $\hat{s}$, we obtain the fidis of the parabolic growth process Ψ defined in Section 3. Let $\mathbb{N}$ denote the positive integers.

Proposition 5.1 *Let $n \in \mathbb{N}$, $0 \leq v_1 < v_2 < \cdots < v_n$, and $h_i \in (0, \infty)$ for all $i = 1, ..., n$. Then*

$$\lim_{\lambda \to \infty} P[\lambda^{2/3} s(v_1, \mathcal{P}_\lambda) \geq h_1; \cdots; \lambda^{2/3} s(v_n, \mathcal{P}_\lambda) \geq h_n] = \prod_{k=1}^{n} \exp\left\{-\frac{4\sqrt{2}}{3} h_k^{3/2}\right\}$$

and

$$\lim_{\lambda\to\infty} P[\lambda^{2/3}s(\lambda^{-1/3}v_1,\mathcal{P}_\lambda)\ge h_1;\cdots;\lambda^{2/3}s(\lambda^{-1/3}v_n,\mathcal{P}_\lambda)\ge h_n]$$

$$=\exp\left(-\int_{\inf_{1\le i\le n}(v_i-\sqrt{2h_i})}^{\sup_{1\le i\le n}(v_i+\sqrt{2h_i})}\sup_{1\le i\le n}[(h_i-\frac{1}{2}(u-v_i)^2)\mathbf{1}_{|u-v_i|\le\sqrt{2t_i}}]du\right).$$

Proof. The first assertion is obtained by noticing that the events $\{s(v_1,\mathcal{P}_\lambda)\ge\lambda^{-2/3}h_1\}$ are independent as soon as $h_1\in(0,\frac{\lambda^{2/3}}{2}\min_{1\le k\le n}(1-\cos(v_{k+1}-v_k)))$. We then apply Lemma 5.1 to estimate the probability of each of these events.

For the second assertion, it suffices to determine the area $\ell(\mathcal{D}_n)$ of the domain

$$\mathcal{D}_n:=\bigcup_{1\le i\le n}\mathrm{cap}_\lambda[v_i,h_i].$$

This set is contained in the angular sector between $\alpha_n:=\inf_{1\le i\le n}[\lambda^{-1/3}v_i-\arccos(1-\lambda^{-2/3}h_i)]$ and $\beta_n:=\sup_{1\le i\le n}[\lambda^{-1/3}v_i+\arccos(1-\lambda^{-2/3}h_i)]$. Denote by $\rho_n(\cdot)$ the radial function which associates to θ the distance between the origin and the point in $\mathcal{D}_n$ closest to the origin lying on the half-line making angle θ with the positive x-axis. Then

$$\begin{aligned}\ell(\mathcal{D}_n) &= \int_{\alpha_n}^{\beta_n}\frac{1}{2}(1-\rho_n^2(\theta))d\theta\\ &= \lambda^{-1/3}\int_{\lambda^{1/3}\alpha_n}^{\lambda^{1/3}\beta_n}\frac{1}{2}(1-\rho_n^2(\lambda^{-1/3}u))du\\ &\underset{\lambda\to\infty}{\sim} \lambda^{-1/3}\int_{\inf_{1\le i\le n}(v_i-\sqrt{2h_i})}^{\sup_{1\le i\le n}(v_i+\sqrt{2h_i})}(1-\rho_n(\lambda^{-1/3}u))du.\end{aligned}$$

Here and elsewhere in this section the terminology $f(\lambda)\underset{\lambda\to\infty}{\sim}g(\lambda)$ signifies that $\lim_{\lambda\to\infty}f(\lambda)/g(\lambda)=1$. Each set $\mathrm{cap}_\lambda[v_i,h_i]$ is bounded by a line with the polar equation

$$\rho=\frac{1-\lambda^{-2/3}h_i}{\cos(\theta-\lambda^{-1/3}v_i)}.$$

Consequently, the function $\rho_n(\cdot)$ satisfies for every $\theta\in(0,2\pi)$,

$$1-\rho_n(\theta)=\sup_{1\le i\le n}\left[\frac{\cos(\theta-\lambda^{-1/3}v_i)-1+\lambda^{-2/3}h_i}{\cos(\theta-\lambda^{-1/3}v_i)}\mathbf{1}_{|\theta-\lambda^{-1/3}v_i|\le\arccos(1-\lambda^{-2/3}h_i)}\right].$$

It remains to determine the asymptotics of the above function. We obtain that

$$1-\rho_n(\lambda^{-1/3}u)\underset{\lambda\to\infty}{\sim}\lambda^{-2/3}\sup_{1\le i\le n}\left[(h_i-\frac{1}{2}(u-v_i)^2)\mathbf{1}_{|u-v_i|\le\sqrt{2h_i}}\right].$$

Considering that the required probability is equal to $\exp(-\lambda\ell(\mathcal{D}_n))$, we complete the proof. $\square$

Remark 1. Proposition 5.1 could have been obtained through the use of the growth process Φ. Indeed, we have $\partial\Phi(v_i)$ greater than h_i for every $1 \leq i \leq n$ iff none of the points (v_i, h_i) is covered by a parabola of Φ. Equivalently, this means that there is no point of $\mathcal{P}$ in the region arising as union of translated downward parabolae $\Pi^{\downarrow}$ with peaks at (v_i, h_i). Calculating the area of this region yields Proposition 5.1.

5.2 The process $\hat{r}_\lambda$

This subsection, devoted to distributional results for $\hat{r}_\lambda$, follows the same lines as the previous one. The problem of determining the distribution of $r(v_0, \mathcal{P}_\lambda)$ seems to be a bit more tricky. To proceed, we fix a direction v_0 and a point $x = (1-h)u_{v_0}$ ($h \in [0,1]$) inside the unit-disk (see Figure 3). Consider an angular sector centered at x and opening away from the origin. Open the sector until it first meets a point of the Poisson point process at the angle $\mathcal{A}_{\lambda,h}$ (the set with dashed lines must be empty in Figure 3). Let $\mathcal{A}_{\lambda,h}$ be the minimal angle of opening from $x = (1-h)u_{v_0}$ in order to meet a point of $\mathcal{P}_\lambda$ in the opposite side of the origin. In particular, when $\mathcal{A}_{\lambda,h} = \alpha$, there is no point of $\mathcal{P}_\lambda$ in

$$\mathcal{S}_{\alpha,h} := \{y \in \mathbb{D}; \langle y - x, u_{v_0}\rangle \geq \cos(\alpha)|y - x|\}.$$

Consequently, we have

$$P[\mathcal{A}_{\lambda,h} \geq \pi/2] = P[s(\theta_0, \mathcal{P}_\lambda) \geq h]. \tag{5.2}$$

The next lemma provides the distribution of $\mathcal{A}_{\lambda,h}$.

Lemma 5.2 *For every $0 \leq \alpha \leq \pi/2$ and $h \in [0,1]$, we have*

$$P[\mathcal{A}_{\lambda,h} \geq \alpha] = \exp\{-\lambda\ell(\mathcal{S}_{\alpha,h})\} \tag{5.3}$$

with

$$\ell(\mathcal{S}_{\alpha,h}) = \left(\alpha + \frac{(1-h)^2}{2}\sin(2\alpha) - (1-h)\sin(\alpha)\sqrt{1-(1-h)^2\sin^2(\alpha)} - \arcsin((1-h)\sin(\alpha))\right). \tag{5.4}$$

When λ goes to infinity, $\mathcal{A}_{\lambda,\lambda^{-2/3}h}$ converges in distribution to a measure with mass 0 on $[0,\pi/2[$ and mass $(1-\exp\{-\frac{4\sqrt{2}}{3}h^{2/3}\})$ on $\{\pi/2\}$.

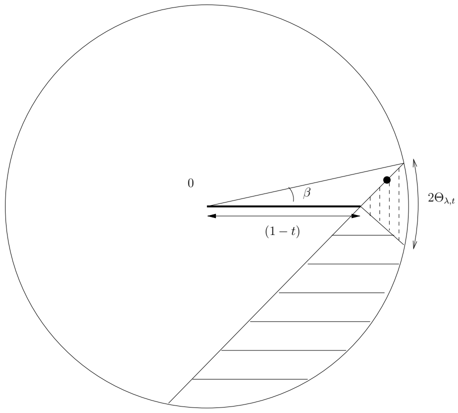

Figure 3: When is a point included in the convex hull?

Proof. A quick geometric consideration shows that the set $\mathcal{S}_{\alpha,h}$ is seen from the origin with an angle equal to

$$2\beta = 2[\alpha - \arcsin((1-h)\sin(\alpha))]. \tag{5.5}$$

The equality (5.4) is then obtained by integrating in polar coordinates:

$$\begin{aligned}
\ell(\mathcal{S}_{\alpha,h}) &= 2\int_0^\beta \left[\int_{\frac{\sin(\alpha-\gamma)}{\sin(\alpha-\theta)}}^1 \rho d\rho\right] d\theta \\
&= \int_0^\beta \left(1 - \frac{(1-h)^2\sin^2(\alpha)}{\sin^2(\alpha-\theta)}\right) d\theta \\
&= \beta - (1-h)^2\sin^2(\alpha)\left(\frac{1}{\tan(\alpha-\beta)} - \frac{1}{\tan(\alpha)}\right)
\end{aligned}$$

and we conclude with the use of (5.5).

Let us show now the last assertion. Using Proposition 5.1 and (5.2), we get that

$$\lim_{\lambda\to\infty} P[\mathcal{A}_{\lambda,\lambda^{-2/3}h} \geq \pi/2] = \exp\left(-\frac{4\sqrt{2}}{3}h^{2/3}\right).$$

It remains to remark that for every $\alpha < \pi/2$, $\lim_{\lambda\to\infty} P[\mathcal{A}_{\lambda,\lambda^{-2/3}h} \geq \alpha] = 1$. Indeed, a direct expansion in (5.4) shows that

$$\ell(\mathcal{S}_{\alpha,\lambda^{-2/3}h}) \underset{\lambda\to\infty}{\sim} \left(\sin(\alpha)\cos(\alpha) + 2\frac{\sin^3(\alpha)}{\cos(\alpha)} - \frac{\sin^3(\alpha)}{2\cos^3(\alpha)}\right)\lambda^{-4/3}h^2.$$

Inserting this estimation in (5.3) completes the proof. ∎

The next lemma provides the explicit distribution of $r(\theta_0, \mathcal{P}_\lambda)$ in terms of $\mathcal{A}_{\lambda,h}$.

Lemma 5.3 *For every $h \in [0,1]$,*

$$P[r(v_0,\mathcal{P}_\lambda) \geq h] = P[s(v_0,\mathcal{P}_\lambda) \geq h] + \lambda \int_0^{\pi/2} \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha} \exp\{-\lambda \ell(\mathrm{cap}_1[v_0,(1-(1-h)\sin(\alpha))])\} d\alpha \tag{5.6}$$

where $\ell(\mathrm{cap}_1[v_0,(1-(1-h)\sin(\alpha))])$ and $\ell(\mathcal{S}_{\alpha,h})$ are defined at (5.1) and (5.4), respectively.

Proof. For fixed $h \in [0,1]$ and $\alpha \in [0,\pi/2)$, we define the set (which is hatched in Figure 3)

$$\mathcal{F}_{h,\alpha} := \mathrm{cap}_1[(\theta_0 - \frac{\pi}{2} + \alpha),(1-(1-h)\sin(\alpha))] \setminus \mathcal{S}_{\alpha,h}.$$

We remark that x is outside the convex hull if and only if either $\mathcal{A}_{\lambda,h}$ is bigger than $\pi/2$ or $\mathcal{F}_{h,\alpha}$ is empty. Consequently, we have

$$P[r(\theta_0,\mathcal{P}_\lambda) \geq h] = P[\mathcal{A}_{\lambda,h} \geq \pi/2] + \int_0^{\pi/2} \exp\{-\lambda \ell(\mathcal{F}_{h,\alpha})\} dP_{\mathcal{A}_{\lambda,h}}(\alpha)$$

where dP_X denotes the distribution of X. Applying Lemma 5.2 yields the result. □

The next proposition provides the asymptotic behavior of the distribution of $\widehat{r}_\lambda(v_0)$:

Proposition 5.2 *We have for every $v, h \geq 0$,*

$$\lim_{\lambda\to\infty} P[\lambda^{2/3} r(v_0,\mathcal{P}_\lambda) \geq h] = \exp\left\{-\frac{4\sqrt{2}h^{3/2}}{3}\right\} + 2\int_0^\infty \exp\left\{-\frac{4\sqrt{2}}{3}(h+\frac{u^2}{2})^{3/2}\right\} u^2 du - 1.$$

Proof. We focus on the asymptotic behavior of the integral in the relation (5.6) where h is replaced with $\lambda^{-2/3}h$. We proceed with the change of variable $\alpha = \frac{\pi}{2} - \lambda^{-1/3}u$:

$$\begin{aligned} &\lambda \int_0^{\pi/2} \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha}(\alpha, \lambda^{-2/3}h) \exp\{-\lambda \ell(\mathrm{cap}_1[v_0,(1-(1-\lambda^{-2/3}h)\sin(\alpha))])\} d\alpha \\ &= \lambda^{2/3} \int_0^{\frac{\pi}{2}\lambda^{1/3}} \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha}(\frac{\pi}{2} - \lambda^{-1/3}u, \lambda^{-2/3}) \\ &\qquad \exp\{-\lambda \ell(\mathrm{cap}_1[v_0,(1-(1-\lambda^{-2/3}h)\cos(\lambda^{-1/3}u))])\} d\alpha. \end{aligned} \tag{5.7}$$

Using (5.1), we find the exponential part of the integrand, which yields

$$\lim_{\lambda\to\infty} \exp\{-\lambda \ell(\mathrm{cap}_1[v_0,(1-(1-\lambda^{-2/3}h)\sin(\frac{\pi}{2}-\lambda^{-1/3}u))])\} = \exp\left\{-\frac{4\sqrt{2}}{3}\left(h+\frac{u^2}{2}\right)^{3/2}\right\}. \tag{5.8}$$

Moreover, the derivative of the area of $\mathcal{S}_{\alpha,h}$ is

$$\frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha} = 1 + (1-h)^2\cos(2\alpha) - 2(1-h)\cos(\alpha)\sqrt{1-(1-h)^2\sin^2(\alpha)}.$$

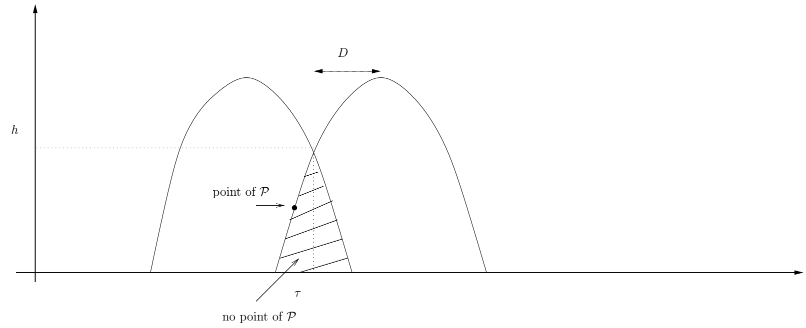

Figure 4: Definition of the r.v. D

In particular, we have

$$\frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha}(\frac{\pi}{2} - \lambda^{-1/3}u, \lambda^{-2/3}h) \underset{\lambda\to\infty}{\sim} 2\lambda^{-2/3}\left[h + u^2 - u\sqrt{2h+u^2}\right]. \tag{5.9}$$

Inserting (5.8) and (5.9) into (5.7) and using (5.6), we obtain the required result. $\square$

Remark 2. In connection with Section 3, the above calculation could have been alternatively based on the limiting hull process related to $\widehat{r}$. Indeed, for fixed $v_0, h \in \mathbb{R}_+$, saying that $\partial\Psi(v_0)$ is greater than h means that there is no translate of the standard downward parabola $\Pi^\downarrow$ containing two extreme points on its boundary and lying underneath the point (v_0, h). We define a random variable D related to the point (v_0, h) (see Figure 4). If $\mathcal{P} \cap ((v_0, h) \oplus \Pi^\downarrow)$ is empty, then we take $D = 0$. Otherwise, we consider all the translates of $\Pi^\downarrow$ containing on the boundary at least one point from $\mathcal{P} \cap ((v_0, h) \oplus \Pi^\downarrow)$ and the point (v_0, h). There is almost surely precisely one among them which has the farthest peak (with respect to the first coordinate) from (v_0, h). The random variable D is then defined as the difference between the *v-coordinate* of the farthest peak and v_0. The distribution of $|D|$ can be made explicit:

$$P[|D| \le t] = \exp\left\{-\frac{2}{3}(2h+t^2)^{3/2} + t(2h + \frac{2}{3}t^2)\right\}, \quad t \ge 0.$$

Conditionally on $|D|$, $\partial\Psi(v_0)$ is greater than h iff the region between the v-axis and the parabola with the farthest peak does not contain any point of $\mathcal{P}$ in its interior. Consequently, we have

$$\begin{aligned} P[\partial\Psi(v_0) \ge h] &= P[D = 0] \\ &\quad + \int_0^\infty \exp\left\{\left(-\frac{4\sqrt{2}}{3}(h + \frac{v^2}{2})^{3/2} - \frac{2}{3}(2h+v^2)^{3/2} - v(2h + \frac{2}{3}v^2)\right)\right\} dP_{|D|}(v), \end{aligned}$$

which provides exactly the result of Proposition 5.2.

The final proposition is the analogue of Proposition 5.1 where the radius-vector function of the flower is replaced by the one of the convex hull itself.

Proposition 5.3 *Let $n \in \mathbb{N}$, $0 \le v_1 < v_2 < \cdots < v_n$ and $h_i \in (0,\infty)$ for all $i = 1, ..., n$. Then*

$$P[\lambda^{2/3} r(v_1, \mathcal{P}_\lambda) \ge h_1; \cdots ; \lambda^{2/3} r(v_n, \mathcal{P}_\lambda) \ge h_n] \underset{\lambda\to\infty}{\sim} \prod_{i=1}^{n} P[\lambda^{2/3} r(v_i, \mathcal{P}_\lambda) \ge h_i]$$

and

$$\lim_{\lambda\to\infty} P[\lambda^{2/3} r(\lambda^{-1/3} v_1, \mathcal{P}_\lambda) \ge h_1; \cdots ; \lambda^{2/3} r(\lambda^{-1/3} v_n, \mathcal{P}_\lambda) \ge h_n]$$

$$= \int_{\mathbb{R}^n} \exp\{-F((t_i, h_i, v_i)_{1\le i\le n})\}\, dP_{(D_1,\cdots,D_n)}(t_1, \cdots, t_n)$$

where $D_1, \cdots, D_n$ are symmetric variables such that

$$P[|D_1| \le t_1; \cdots ; |D_n| \le t_n] = \exp\left(-\int \sup_{1\le i\le n} [(h_i + \frac{t_i^2}{2} - \frac{(|v - v_i| + t_i)^2}{2})) \vee 0] dv \right) \tag{5.10}$$

and F is the area

$$F((t_i, h_i, v_i)_{1\le i\le n}) = \int_{\mathbb{R}} \left\{ \sup_{1\le i\le n} \left[\left(h_i + \frac{t_i^2}{2} - \frac{(v - v_i - t_i)^2}{2} \right) \vee 0 \right] \right.$$
$$\left. - \sup_{1\le i\le n} \left[\left(h_i + \frac{t_i^2}{2} - \frac{(|v - v_i| + t_i)^2}{2} \right) \vee 0 \right] \right\} dv. \tag{5.11}$$

Proof. We prove the first assertion and denote by $\mathcal{A}_1, \cdots, \mathcal{A}_n$ the angles (as defined by Lemma 5.2) corresponding to the couples $(v_1, \lambda^{-2/3} h_1), \cdots, (v_n, \lambda^{-2/3} h_n)$. Conditionally on $\{\mathcal{A}_i = \alpha_i\}$, the event $\{\lambda^{2/3} r(v_i, \mathcal{P}_\lambda) \ge h_i\}$ only involves the points of the point process $\mathcal{P}_\lambda$ included in the circular cap $\mathrm{cap}_1[v_i - \frac{\pi}{2} + \alpha_i, (1 - (1 - \lambda^{-2/3} h_i) \sin(\alpha_i))]$ (see the proof of Lemma 5.3). Moreover there exists $\delta \in (0, \pi/2)$ such that for λ large enough and $\alpha_i \in (\delta, \frac{\pi}{2})$ for every i, these circular caps are all disjoint. Consequently, we obtain that conditionally on $\{\mathcal{A}_i > \delta\ \forall i\}$ the events $\{\lambda^{2/3} r(v_i, \mathcal{P}_\lambda) \ge h_i\}$ are independent. It remains to remark that Lemma 5.2 implies

$$\lim_{\lambda\to\infty} P[\exists 1 \le i \le n;\ \mathcal{A}_i \le \delta] = 0.$$

Let us consider now the second assertion, which could be obtained by a direct estimation of the joint distribution of the angles $\mathcal{A}_i$ (corresponding to the points $(\lambda^{-1/3} v_i, \lambda^{-2/3} h_i)$). But it is easier to prove it with the use of the boundary $\partial\Psi$ of the hull process. As in Remark 2, we define for each

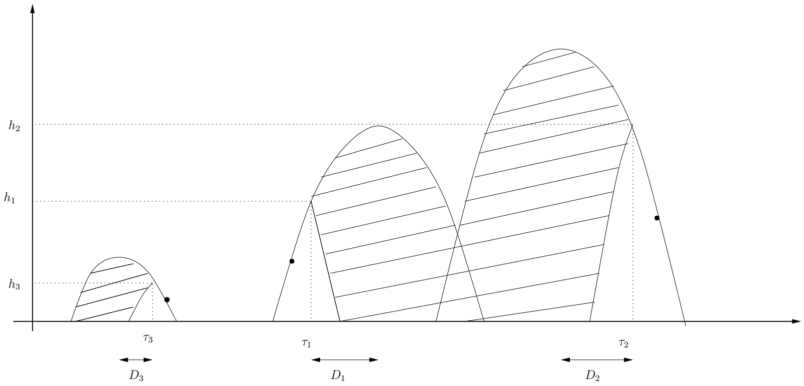

Figure 5: Definition of the area F (hatched region). The black points belong to $\mathcal{P}$.

point (v_i, h_i), the random variable D_i as the difference between the v-coordinate of the farthest peak of a downward parabola arising as a translate of $\Pi^{\downarrow}$ (denoted by Par_i) containing on its boundary (v_i, h_i) and a point of $\mathcal{P}$. Then $|D_i|$ is less than t_i for every $1 \leq i \leq n$ iff there is no point of $\mathcal{P}$ inside a region delimited by the v-axis and the supremum of n functions $g_1, \cdots, g_n$ defined in the following way: $g_i(v_i + \cdot)$ is an even function with a support equal to $[t_i - \sqrt{2h_i + t_i^2}, t_i + \sqrt{2h_i + t_i^2}]$ and identified with the parabola $\mathrm{Par}_i(\cdot - v_i)$ on the segment $[t_i - \sqrt{2h_i + t_i^2}, 0]$ (see Figure 4). We deduce from this assertion the result (5.10). Conditionally on $\{D_1 = t_1, \cdots, D_n = t_n\}$, $\partial\Psi(v_i)$ is greater than h_i for every i iff the region between the functions g_i and the parabolae Par_i does not contain any point of $\mathcal{P}$ (see Figure 5). This implies the result (5.11) and completes the proof. $\square$

Remark 3. Convergence of the fidis of the radius-vector function of the convex hull of n uniform points in the disk has already been derived in Theorem 2.3 of [19]. Still we feel that the results presented in this section are obtained in a more direct and explicit way. Moreover, they are characterized by the parabolic growth and hull processes, which provides an elementary representation of the asymptotic distribution.

The explicit fidis and the convergence to those of $\partial\Psi$ and $\partial\Phi$ can be used to obtain explicit formulae for second-order characteristics of the point process of extremal points. Details are postponed to the appendix.

6 Stabilizing functional representation for convex hull characteristics

The purpose of this section is to link the convex hull characteristics considered in Section 1 with the theory of stabilizing functionals, a convenient tool for proving limit theorems in geometric contexts, see [6] [25]-[30].

Define the following geometric functionals, often referred to as the basic functionals in the sequel, as opposed to the scaling limit functionals discussed below.

- The point-configuration functional $\xi_s(x, \mathcal{X}),\ x \in \mathcal{X} \subset \mathbb{B}_d$, for finite $\mathcal{X} \subset \mathbb{B}_d$ is set to be zero if x is not a vertex of $\mathrm{conv}(\mathcal{X})$ and otherwise it is defined as follows. Let $\mathcal{F}(x, \mathcal{X})$ be the (possibly empty) collection of faces f in $\mathcal{F}_{d-1}(\mathrm{conv}(\mathcal{X}))$ such that $x = \mathrm{Top}(f)$, where $\mathrm{Top}(f)$ is at (2.5). Let $\mathrm{cone}(\mathcal{F}(x, \mathcal{X})) := \{ry,\ r > 0,\ y \in \mathcal{F}(x, \mathcal{X})\}$ be the corresponding cone. Given $x \in \mathrm{ext}(\mathcal{X})$, define $\xi_s(x, \mathcal{X})$ to be $\mathrm{Vol}([\mathbb{B}_d \setminus F(\mathcal{X})] \cap \mathrm{cone}(\mathcal{F}(x, \mathcal{X})))$. Recall that $F(\cdot)$ is the Voronoi flower defined at (2.3). By (2.28),

$$H_\lambda^{\xi_s}(v) := \sum_{x \in \mathcal{P}_\lambda,\ x/|x| \in \exp([\mathbf{0}, v])} \xi_s(x; \mathcal{P}_\lambda),\ v \in \mathbb{R}^{d-1}, \tag{6.1}$$

  is asymptotic to $W_\lambda(v)$ as $\lambda \to \infty$ and the same holds for the centered versions of these random variables, hence the notation ξ_s because W_λ arises in (2.28) as a suitable integral of s_λ. By '*asymptotic to*' we mean expressions differing by lower order terms as $\lambda \to \infty$ negligible in our asymptotic argument, the so understood asymptotic expressions can be safely interchanged with one another in the proofs of our asymptotic statements. Here and elsewhere, expressions termed 'asymptotic' differ only due to boundary effects, themselves rendered negligible by stabilization; we omit the formal definition of 'boundary effects' and the related conceptually trivial but technically tedious considerations establishing their negligibility. Recalling (2.30) we have

$$\sum_{x \in \mathcal{P}_\lambda} \xi_s(x; \mathcal{P}_\lambda) = W_\lambda. \tag{6.2}$$

- Likewise, we define $\xi_r(x; \mathcal{X}),\ x \in \mathcal{X} \subset \mathbb{B}_d$ to be $\mathrm{Vol}([\mathbb{B}_d \setminus \mathrm{conv}(\mathcal{X})] \cap \mathrm{cone}(\mathcal{F}(x, \mathcal{X})))$ if $x \in \mathrm{ext}(\mathcal{X})$ and we set $\xi_r(x; \mathcal{X}) := 0$ otherwise. It is clear by (2.29) that

$$H_\lambda^{\xi_r}(v) := \sum_{x \in \mathcal{P}_\lambda,\ x/|x| \in \exp([\mathbf{0}, v])} \xi_r(x; \mathcal{P}_\lambda) \tag{6.3}$$

is asymptotic to $V_\lambda(v)$, $v \in \mathbb{R}^{d-1}$, and the same holds for the centered versions of these random variables. By (2.30) we have

$$\sum_{x \in \mathcal{P}_\lambda} \xi_r(x; \mathcal{P}_\lambda) = V_\lambda. \tag{6.4}$$

- The k-th order projection avoidance functional $\xi_{\vartheta_k}(x; \mathcal{X})$, $x \in \mathcal{X}$, with $k \in \{1, ..., d\}$ is zero if $x \notin \mathrm{ext}(\mathcal{X})$, and otherwise equal to

$$\xi_{\vartheta_k}(x; \mathcal{X}) := \int_{[\mathbb{B}_d \setminus \mathrm{conv}(\mathcal{X})] \cap \mathrm{cone}(\mathcal{F}(x,\mathcal{X}))} \frac{1}{|x|^{d-k}} \vartheta_k^{\mathcal{X}}(x) dx,$$

  see (2.14). In particular, (2.15) yields

$$V_k(B_d) - V_k(K_\lambda) = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} [\sum_{x \in \mathcal{P}_\lambda} \xi_{\vartheta_k}(x; \mathcal{P}_\lambda)]. \tag{6.5}$$

- The k-face functional $\xi_{f_k}(x; \mathcal{X})$, defined for finite $\mathcal{X}$ in $\mathbb{B}_d$, $x \in \mathcal{X}$, and $k \in \{0, 1, ..., d-1\}$, is the number of k-dimensional faces f of $\mathrm{conv}(\mathcal{X})$ with $x = \mathrm{Top}(f)$, if x belongs to Vertices[conv($\mathcal{X}$)], and zero otherwise. Thus $\sum_{x \in \mathcal{X}} \xi_{f_k}(x; \mathcal{X})$ is the total number of k-faces in $\mathrm{conv}(\mathcal{X})$. In particular, setting $\mathcal{X} := \mathcal{P}_\lambda$, the total mass of $\mu_\lambda^{f_k}$ is

$$f_k(K_\lambda) = \sum_{x \in \mathcal{P}_\lambda} \xi_{f_k}(x; \mathcal{P}_\lambda). \tag{6.6}$$

  It is readily seen by definition (2.5) of $\mu_\lambda^{f_k}$ that

$$\mu_\lambda^{f_k} = \mu_\lambda^{\xi_{f_k}} := \sum_{x \in \mathcal{P}_\lambda} \xi_{f_k}(x; \mathcal{P}_\lambda) \delta_x. \tag{6.7}$$

A curious reader might wonder why we do not consider functionals defined directly in terms of the local curvature measures Φ_k. The reason is that under their natural scaling they would give rise to the so-called *vanishing add-one cost* in terms of stabilization theory, resulting in vanishing asymptotic variance. Consequently, instead of the central limit theorem aimed at, we would only obtain a convergence in probability to zero under lower order scaling. This non-trivial phenomenon falls beyond the scope of this paper though.

In the spirit of the local scaling Section 4, we shall construct *scaling counterparts* of the above functionals defined in terms of the paraboloid growth and hull processes. To reflect this correspondence we write $\xi_\cdot^{(\infty)}$ marking the local scaling limit analog of $\xi_\cdot$ with the (∞) superscript.

- The functional $\xi_s^{(\infty)}(x;\mathcal{P})$ is defined to be zero if $x \notin \text{ext}(\Psi)$ and otherwise is defined as follows. Let $\mathcal{F}^\infty(x,\mathcal{P})$ stand for the collection of paraboloid faces f of Φ for which $x = \text{Top}(f)$ (recall (2.5)) and let v-cone$(\mathcal{F}^\infty(x,\mathcal{P}))$ be the cylinder (vertical cone) in $\mathbb{R}^{d-1}\times\mathbb{R}_+$ generated by $\mathcal{F}^\infty(x,\mathcal{P})$, that is to say v-cone$(\mathcal{F}^\infty(x,\mathcal{P})) := \{(v,h),\ \exists\, h' :\ (v,h') \in \mathcal{F}^\infty(x,\mathcal{P})\}$. Then, if $x \in \text{ext}(\Psi)$, we set $\xi_s^{(\infty)}(x;\mathcal{P}) := \text{Vol}(\text{v-cone}(\mathcal{F}^\infty(x,\mathcal{P})) \setminus \Psi)$. Formally we should define $\xi_s^{(\infty)}(x;\mathcal{X})$ for general $\mathcal{X}$ rather than just for $\mathcal{P}$, but we avoid it not to introduce extra notation since we will mainly consider $\mathcal{X} = \mathcal{P}$ anyway, yet the general definition can be readily recovered by formally conditioning on $\mathcal{P} = \mathcal{X}$. This simplifying convention will also be applied for the remaining local scaling functionals below.
- Likewise, $\xi_r^{(\infty)}(x;\mathcal{P})$ is zero if $x \notin \text{ext}(\Psi)$ and otherwise $\xi_r^{(\infty)}(x;\mathcal{P}) := \text{Vol}(\text{v-cone}(\mathcal{F}^\infty(x,\mathcal{P}))\setminus \Phi)$.
- The k-th order projection avoidance functional $\xi_{\vartheta_k}^{(\infty)}(x;\mathcal{P})$ is zero if $x \notin \text{ext}(\Psi)$ and otherwise

$$\xi_{\vartheta_k}^{(\infty)} := \int_{\text{v-cone}(\mathcal{F}^\infty(x,\mathcal{P}))\setminus\Phi} \vartheta_k^\infty(x)dx \tag{6.8}$$

  with $\vartheta_k^\infty(\cdot)$ defined in (3.23). Note that the extra factor $\frac{1}{|x|^{d-k}}$ in (2.15) converges to one under the scaling T^λ and thus is not present in the asymptotic functional.
- The k-face functional $\xi_{f_k}^{(\infty)}(x;\mathcal{P})$, defined for $x \in \mathcal{P}$, and $k \in \{0,1,...,d-1\}$, is the number of k-dimensional paraboloid faces f of the hull process Φ for which $x = \text{Top}(f)$, if x belongs to $\text{ext}(\Psi) = \text{Vertices}(\Phi)$, and zero otherwise.

For each basic functional, with generic notation ξ, we consider its finite-size scaling counterpart

$$\xi^{(\lambda)}(x,\mathcal{X}) := \xi([T^\lambda]^{-1}x, [T^\lambda]^{-1}\mathcal{X}),\ x \in \mathcal{X} \subset \mathcal{R}_\lambda \subset \mathbb{R}^{d-1}\times\mathbb{R}_+ \tag{6.9}$$

where T^λ is the scaling transform (2.19) and $\mathcal{R}_\lambda$ its image (2.20). Again resorting to the theory developed in Section 4 we see that $\xi^{(\lambda)}$ can be regarded as *interpolating* between ξ and $\xi^{(\infty)}$. However, due to the differing natures of the functionals considered here, different scaling prefactors are needed to ensure non-trivial scaling behaviors. More precisely, for each ξ discussed above we define its *proper scaling prefactor* $\lambda^{\eta[\xi]}$ where

- $\eta[\xi_s] = \eta[\xi_r] = \eta[\xi_{\vartheta_k}] = \beta(d-1)+\gamma,\ k \in \{0,1,...,d-1\}$, due to the scaling corresponding to that applied in Theorem 4.1 (note that the re-scaled projection avoidance function (2.27) involves no scaling prefactor).

- $\eta[\xi_{f_k}] = 0$ because the number of k-faces does not undergo any scaling.

To proceed, for any measurable $D \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ and generic scaling limit functional $\xi^{(\infty)}$, by its *restricted version* we mean by $\xi_D^{(\infty)}(x;\mathcal{P}) := \xi^{(\infty)}(x;\mathcal{P} \cap D)$.Note that the so-defined restricted functionals in case of D bounded, or of bounded spatial extent, clearly involve growth and hull processes built on input of bounded spatial extent, in which case the definition (3.6) for $\mathcal{P}$ replaced with $\mathcal{P} \cap D$ yields infinite vertical faces at the boundary of $D's$ spatial extent. This makes the functionals considered in this paper infinite or even undefined for points close to these infinite faces. However, this *boundary pathology* is not a problem when considering the values of functional at a given point x, with the set D containing neighborhoods of x whose size tends to infinity, as is always the case in our argument below. Indeed, the *boundary pathology* is then *pushed away* from x and only occurs with negligible probabilities, and thus has no effect on the asymptotic theory. Keeping this in mind we simply let our functionals assume some default value, say 0, whenever infinite or undefined, and keep working in our standard setting, rather than introducing a clumsy and technical notion of a *partially defined geometric functional*.

Recalling that $B_{d-1}(v,r)$ is the $(d-1)$ dimensional ball centered at $v \in \mathbb{R}^{d-1}$ with radius r, let $C_{d-1}(v,r)$ be the cylinder $B_{d-1}(v,r) \times \mathbb{R}_+$. Given a generic scaling limit functional $\xi^{(\infty)}$, we shall write $\xi_{[r]}^{(\infty)} := \xi_{C_{d-1}(v,r)}^{(\infty)}$. Likewise, for the finite scaling functionals $\xi^{(\lambda)}$ we shall use the notation $\xi_{[r]}^{(\lambda)}$ with a fully analogous meaning.

A random variable $R := R^{\xi^{(\infty)}}[x]$ is called a *localization radius* for the functional $\xi^{(\infty)}$ iff a.s.

$$\xi^{(\infty)}(x;\mathcal{P}) = \xi_{[r]}^{(\infty)}(x;\mathcal{P}) \text{ for all } r \geq R.$$

We analogously define the notion of a localization radius for $\xi^{(\lambda)}$. The notion of localization, developed in [40], is a variant of a general concept of stabilization [29, 30, 6]. A crucial property of the functionals $\xi^{(\lambda)}$ and $\xi^{(\infty)}$ is that they admit localization radii with rapidly decaying tails.

Lemma 6.1 *The functionals $\xi^{(\infty)}$ and $\xi^{(\lambda)}$ admit localization radii with the property that*

$$P[R > L] \leq C \exp(-L^{d+1}/C) \tag{6.10}$$

for some finite positive constant C, uniformly in λ large enough and uniformly in x.

Proof. The proof is given for the scaling limit functionals $\xi^{(\infty)}$ only; the argument for the finite scaling functionals $\xi^{(\lambda)}$ is fully analogous.

For a point $x = (v, h) \in \mathcal{P}$ denote by $\mathcal{P}[[x]]$ the collection of all vertices of $(d-1)$-dimensional faces of Φ meeting at x if $x \in \mathrm{Vertices}(\Phi)$ and $\mathcal{P}[[x]] := \{x\}$ otherwise. If $x \in \mathrm{Vertices}(\Phi)$, the collection $\mathcal{P}[[x]]$ uniquely determines the *local facial structure* of Φ at x, understood as the collection of all $(d-1)$-dimensional faces $f_1[x], \ldots, f_m[x]$, $m = m[x] < \infty$ meeting at x. We shall show that there exists a random variable $R' := R'[x]$ with the properties that

- With probability one the facial structure $\mathcal{P}_{[r]}[[x]]$ at x determined upon restricting $\mathcal{P} := \mathcal{P} \cap C_{\mathbb{R}^{d-1}}(v, r)$ coincides with $\mathcal{P}[[x]]$ for all $r \geq R'$; in the sequel we say that $\mathcal{P}[[x]]$ is fully determined within radius R' in such a case.

- We have
$$P[R' > L] \leq C \exp(-L^{d+1}/C). \tag{6.11}$$

Before proceeding, we note that to conclude the statement of Lemma 6.1, it is enough to establish (6.11). Indeed, this is because

- The values of functionals $\xi_s^{(\infty)}$, ξ_r^* and $\xi_{f_k}^{(\infty)}$, $k \in \{0, \ldots, d-1\}$, at $x \in \mathcal{P}$ are uniquely determined given $\mathcal{P}[[x]]$ and thus R' can be taken as the localization radius.

- The values of functionals $\xi_{\vartheta_k}^{(\infty)}(x, \mathcal{P})$, $k \in \{1, \ldots, d-1\}$, $x = (v, h)$, are determined given the intersection of the hull process Φ with $\Theta[x] := [\text{v-cone}(\mathcal{F}^{(\infty)}(x, \mathcal{P})) \setminus \Phi] \oplus \Pi^{\downarrow}$, see (3.23) and the definition of $\xi_{\vartheta_k}^{(\infty)}$. It is readily seen that this intersection $\Theta[x] \cap \Phi$ is in its turn uniquely determined by $\Theta[x] \cap \mathrm{Vertices}(\Phi)$. Thus, to know it, it is enough to know the facial structure at x and at all vertices of Φ falling into $\Theta[x]$. To proceed, note that the spatial diameter of $\Theta[x]$ is certainly bounded by $R'[x]$ plus $2\sqrt{2}$ times the square root of the highest height coordinate of $\partial\Phi$ within spatial distance $R'[x]$ from v. Use (4.3) to bound this height coordinate and thus to establish a superexponential bound $\exp(-\Omega(L^{d+1}))$ for tail probabilities of the spatial diameter $R''[x]$ of $\Theta[x]$. Finally, we set the localization radius to be $\max_{y \in \mathrm{Vertices}(\Phi),\, y \in C_{\mathbb{R}^{d-1}}(v, R''[x])} R'[y]$ which is again easily verified to exhibit the desired tail behavior as the number of vertices within $C_{\mathbb{R}^{d-1}}(v, R''[x])$ grows polynomially in $R''[x]$ with overwhelming probability, see Lemma 3.2 in [40].

- The values of $\xi_{\Phi_k}^{(\infty)}$, $k \in \{0, \ldots, d-1\}$ at x are fully determined upon knowing $\bigcup_{y \in \mathcal{P}[[x]]} \mathcal{P}[[y]]$ and thus the localization radius can be taken as $\max_{y \in \mathcal{P}[[x]]} [R'[y] + \mathrm{d}(x, y)]$. The required decay is readily concluded from (6.11) by crudely bounding the cardinality of $\mathcal{P}[[x]]$ with the total number of vertices of Φ within $C_{\mathbb{R}^{d-1}}(v, R'[x])$ which again grows polynomially in $R'[x]$ with overwhelming probability, see again Lemma 3.2 in [40].

To proceed with the proof, suppose first that x is not extreme in Φ. Then, by Lemma 3.1 in [40] and its proof, there exists $R' = R'[x]$ satisfying (6.11) and such that the extremality status of x localizes within radius R'. In this particular case of x not extreme in Φ this also implies localization for $\mathcal{P}[[x]] = \{x\}$. Assume now that x is an extreme point in $\mathcal{P}$. Enumerate the $(d-1)$-dimensional faces meeting x by $f_1, ..., f_m$. The local facial structure $\mathcal{P}[[x]]$ is determined by the parabolic faces of the space-time region $\bigcup_{i \le m} \Pi^{\downarrow}[f_i]$, which by (3.16) is devoid of points from $\mathcal{P}$. Note that this region contains all vertices of $f_1, ..., f_m$ on its upper boundary. Moreover, Poisson points outside this region do not change the status of the faces $f_1, ..., f_m$ as these faces will not be subsumed by larger faces meeting x unless Poisson points lie on the boundary of the hull process, an event of probability zero. It follows that $\mathcal{P}[[x]]$ is fully determined by the point configuration $\mathcal{P} \cap C_{d-1}(v, R')$ where R' is the smallest integer r such that:

$$\bigcup_{i \le m} [\Pi^{\downarrow}[f_i] \cap (\mathbb{R}^{d-1} \times \mathbb{R}_+)] \subset C_{d-1}(v, r). \tag{6.12}$$

To establish (6.11) for R' we note that if R' exceeds L, then, by standard geometry, within distance $O(L^2)$ from x we can find a point x' in $\mathbb{Z}^d$ with the properties that

- the downwards parabolic solid $x' \oplus \Pi^{\downarrow}$ is contained in $\bigcup_{i \le m} \Pi^{\downarrow}[f_i]$ and thus in particular devoid of points of $\mathcal{P}$,
- the spatial diameter (the diameter of spatial projection on $\mathbb{R}^{d-1}$) of $[x' \oplus \Pi^{\downarrow}] \cap (\mathbb{R}^{d-1} \times \mathbb{R}_+)$ does exceed $L/2$.

Since the intensity measure of $\mathcal{P}$ assigns to such $[x' \oplus \Pi^{\downarrow}] \cap (\mathbb{R}^{d-1} \times \mathbb{R}_+)$ mass of order at least $\Omega(L^{d+1})$ (in fact even $\Omega(L^{d+1+2\delta})$, see the proof of Lemma 3.1 in [40] for details in a much more general set-up), the probability of having $x' \oplus \Pi^{\downarrow}$ devoid of points of $\mathcal{P}$ is $\exp(-\Omega(L^{d+1}))$. Since the cardinality of $B_d(x, L^2) \cap \mathbb{Z}^{d-1}$ is bounded by CL^{2d}, Boole's inequality gives

$$P[R' > L] \le CL^{2d} \exp(-L^{d+1}/C)$$

which yields the required inequality (6.11) and thus completes the proof of Lemma 6.1. $\square$

7 Variance asymptotics and Gaussian limits for empirical measures

The purpose of the present section is to take advantage of the above asymptotic embedding of convex polytope characteristics into the general set-up of stabilization theory. To this end, we first need the following moment bound.

Lemma 7.1 *For all scaling limit functionals $\xi^{(\infty)}$ and local scaling functionals $\xi^{(\lambda)}$ considered in Section 6 and for all $p > 0$ we have*

$$\sup_{x \in \mathbb{R}^{d-1}} \mathbb{E}\,[\xi^{(\infty)}(x;\mathcal{P})]^p < \infty \quad \textit{and} \quad \sup_{\lambda} \sup_{x \in \mathcal{R}_\lambda} \mathbb{E}\,[\xi^{(\lambda)}(x;\mathcal{P}^{(\lambda)})]^p < \infty. \tag{7.1}$$

Proof. We only give the proof in the limit case $\xi^{(\infty)}$, the finite scaling case $\xi^{(\lambda)}$ being fully analogous. This is done separately for all functionals considered.

- For $\xi_s^{(\infty)}(x;\mathcal{P})$ and $\xi_r^{(\infty)}(x;\mathcal{P})$ we only consider the case of x extreme, for otherwise both functionals are zero. With $x \in \mathrm{Vertices}(\Phi)$ we make use of (4.3) to bound the height and of (6.11) and (6.12) to bound the spatial size of the regions whose volumes define $\xi_s^{(\infty)}$ and $\xi_r^{(\infty)}$. Since these bounds yield superexponential decay rates on each dimension separately, the volume admits uniformly controllable moments of all orders. Finally, by (6.8), $0 \le \xi_{\vartheta_k}^{(\infty)} \le \xi_r^{(\infty)}$ whence (7.1) follows for ξ_{ϑ_k} as well.

- For $\xi_{f_k}^{(\infty)}(x;\mathcal{P})$ we only consider the case $x \in \mathrm{Vertices}(\Phi)$ and we let $N := N[x]$ be the number of extreme points in $\mathcal{P} \cap C_{d-1}(v, R'[x])$ with R' as in (6.12). Then $\xi_{f_k}^{(\infty)}(x;\mathcal{P})$ is upper bounded by $\binom{N}{k-1}$. By Lemma 3.2 of [40], the probability that a point (v_1, h_1) is extreme in Φ falls off superexponentially fast in h_1, see again (4.3). Consequently, in view of (6.11), $\binom{N}{k-1}$ and thus also $\xi_{f_k}^{(\infty)}(x;\mathcal{P})$ admits finite moments of all orders.

The proof is hence complete. □

This puts us now in a position to apply the general results of stabilization theory and the particular results of [40]. To this end, define for a generic functional ξ

$$\mu_\lambda^\xi := \sum_{x \in \mathcal{P}_\lambda} \xi(x;\mathcal{P}_\lambda)\delta_x \tag{7.2}$$

and $\bar{\mu}_\lambda^\xi := \mu_\lambda^\xi - \mathbb{E}\,\mu_\lambda^\xi$. Following [40], we define the second order correlation functions for $\xi^{(\infty)}$

$$\varsigma_{\xi^{(\infty)}}(x) := \mathbb{E}\,[\xi^{(\infty)}(x;\mathcal{P})]^2, \tag{7.3}$$

and

$$\varsigma_{\xi^{(\infty)}}(x,y) := \mathbb{E}\,[\xi^{(\infty)}(x;\mathcal{P}\cup\{y\})\xi^{(\infty)}(y;\mathcal{P}\cup\{x\})] - \mathbb{E}\,[\xi^{(\infty)}(x;\mathcal{P})]\mathbb{E}\,[\xi^{(\infty)}(y;\mathcal{P})] \tag{7.4}$$

together with the asymptotic variance expression (see (1.7) and (1.8) in [40] and recall that we are working in isotropic regime here corresponding to constant $\rho_0 \equiv 1$ there):

$$\sigma^2(\xi^{(\infty)}) := \int_0^\infty \varsigma_{\xi^{(\infty)}}((\mathbf{0},h))dh + \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{d-1}} \varsigma_{\xi^{(\infty)}}((\mathbf{0},h),(v',h'))h^\delta h'^\delta dh dh' dv'. \tag{7.5}$$

With this notation, in full analogy to Theorem 1.1 and (2.2) in Theorem 2.1 in [40] with $\rho_0 \equiv 1$ there, and in full analogy to Theorems 1.2, 1.3 and (2.3), (2.4) in Theorem 2.1 in [40], using the local scaling results of Section 4 we obtain:

Theorem 7.1 *Let ξ be any of the basic functionals discussed in Section 6. Then, for each $g \in \mathcal{C}(\mathbb{B}_d)$ we get*

$$\lim_{\lambda\to\infty} \lambda^{-\tau}\mathbb{E}\,[\langle g, \lambda^{\eta[\xi]}\mu_\lambda^\xi\rangle] = \int_0^\infty \mathbb{E}\,[\xi^{(\infty)}(\mathbf{0},h)]h^\delta dh \int_{\mathbb{S}_{d-1}} g(x)\sigma_{d-1}(dx). \tag{7.6}$$

Further, the integral in (7.5) converges and for each $g \in \mathcal{C}(\mathbb{B}_d)$ we have

$$\lim_{\lambda\to\infty} \lambda^{-\tau}\mathrm{Var}[\langle g, \lambda^{\eta[\xi]}\mu_\lambda^\xi\rangle] = V^{\xi^{(\infty)}}[g] := \sigma^2(\xi^{(\infty)}) \int_{\mathbb{S}_{d-1}} g^2(x)\sigma_{d-1}(dx) \tag{7.7}$$

with

$$\tau = \beta(d-1) = \frac{d-1}{d+1+2\delta}. \tag{7.8}$$

Furthermore, the random variables $\lambda^{-\tau/2}\langle g, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle$ converge in law to $\mathcal{N}(0, V^{\xi^{(\infty)}}[g])$. Finally, if $\delta = 0$ and $\sigma^2(\xi^{(\infty)}) > 0$, then we have for all $g \in \mathcal{C}(\mathbb{B}_d)$ not identically zero

$$\sup_t \left| P\left[\frac{\langle g, \bar{\mu}_\lambda^\xi\rangle}{\sqrt{\mathrm{Var}[\langle g, \bar{\mu}_\lambda^\xi\rangle]}} \le t\right] - P[\mathcal{N}(0,1) \le t]\right| = O\left(\lambda^{-(d-1)/2(d+1)}(\log\lambda)^{3+2(d-1)}\right). \tag{7.9}$$

Theorem 7.1 follows directly by the methods developed in [40] as quoted above. We will not provide full details of these techniques, as it would involve rewriting large parts of [40] with just minor changes. Nonetheless the objective method for stabilizing functionals, and especially its instance specialized for convex hulls as developed ibidem, may to some extent be regarded as exotic subjects, and so we provide an overview discussion of these techniques in the context of Theorem 7.1 in order to make our paper more self-contained and reader-friendly. As a simple yet representative example we choose the expectation formula (7.6). We begin by writing, for $g \in \mathcal{C}(\mathbb{B}_d)$,

$$\mathbb{E}\left[\langle g, \mu_\lambda^\xi \rangle\right] = \lambda \int_{\mathbb{B}_d} g(x) \mathbb{E}[\xi(x, \mathcal{P}_\lambda)](1 - |x|)^\delta dx. \tag{7.10}$$

Next, for each $x \in \mathbb{B}_d$ consider the version T_x^λ of the scaling transform T^λ given in (2.19), with the transform origin u_0 set there at the radial projection of x onto $\mathbb{S}_{d-1}$. This also gives rise to the x-versions of re-scaled functionals $\xi^{(\lambda;x)}$ as defined in (6.9) and to re-scaled point processes $\mathcal{P}^{(\lambda;x)} := T^{(\lambda;x)}\mathcal{P}_\lambda$. In this language, (7.10) becomes

$$\mathbb{E}\left[\langle g, \mu_\lambda^\xi \rangle\right] = \lambda \int_{\mathbb{B}_d} g(x) \mathbb{E}\left[\xi^{(\lambda;x)}\left((\mathbf{0}, \lambda^\gamma(1 - |x|)), \mathcal{P}^{(\lambda;x)}\right)\right](1 - |x|)^\delta dx. \tag{7.11}$$

Using the rotational invariance of the functionals ξ considered here, we can omit the x in $T^{(\lambda;x)}$ superscript. Thus, putting in addition $h := \lambda^\gamma(1 - |x|)$, we rewrite (7.11) as

$$\mathbb{E}\left[\langle g, \mu_\lambda^\xi \rangle\right] = \lambda \int_{\mathbb{B}_d} g(x) \mathbb{E}\left[\xi^{(\lambda)}\left((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\right)\right] \lambda^{-\gamma\delta} h^\delta dx =$$

$$\lambda^{1-\gamma\delta} \int_{\mathbb{S}_{d-1}} \int_0^{\lambda^\gamma} g((1 - \lambda^{-\gamma}h)u) \mathbb{E}\left[\xi^{(\lambda)}\left((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\right)\right] h^\delta (1 - \lambda^{-\gamma}h)^{d-1} \lambda^{-\gamma} dh \sigma_{d-1}(du).$$

Noting that $\tau = 1 - \delta\gamma - \gamma$ and multiplying through by $\lambda^{-\tau+\eta[\xi]}$ we end up with

$$\lambda^{-\tau+\eta[\xi]}\mathbb{E}\left[\langle g, \mu_\lambda^\xi \rangle\right] = \int_{\mathbb{S}_{d-1}} \int_0^{\lambda^\gamma} g((1-\lambda^{-\gamma})u) \mathbb{E}\left[\lambda^{\eta[\xi]}\xi^{(\lambda)}\left((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\right)\right] (1-\lambda^{-\gamma}h)^{d-1} h^\delta dh \sigma_{d-1}(u). \tag{7.12}$$

This puts us in a position to apply the local scaling results of Section 4 which imply that $\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})$ is a good local approximation for $\lambda^{\eta[\xi]}\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})$ – the exponents $\eta[\xi]$ were chosen so that this be the case. Stabilization properties of ξ yield $\mathbb{E}\,[\lambda^{\eta[\xi]}\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)})] \to \mathbb{E}\,[\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})]$, which is the analog of Lemma 3.3 of [40]. Moreover, $E[\xi^{(\lambda)}((\mathbf{0}, h), \mathcal{P}^{(\lambda)}]h^\delta$ is dominated by an integrable function of h, as the contribution coming from large h is well controllable as in Lemma 3.2 in [40] – in particular we exploit that $\xi(x, \mathcal{X}) = 0$ whenever x is non-extreme in $\mathcal{X}$ and, roughly speaking, only points close enough to the boundary $\mathbb{S}_{d-1}$ have a non-negligible chance of being extreme in $\mathcal{P}_\lambda$. Thus letting $\lambda \to \infty$ in (7.12), using $\lim_{\lambda\to\infty}(1-\lambda^{-\gamma}h)^{d-1} = 1$ and $\lim_{\lambda\to\infty} g((1-\lambda^{-\gamma}h)u) = g(u)$ and applying the dominated convergence theorem as in e.g. Subsection 3.2 in [40], we finally get from (7.12) the required relation (7.6). The proofs of the variance identities (7.7) and Gaussian convergence (7.9) are much more involved, yet again they rely on the same crucial idea of local scaling and stabilization, going together under the name of the objective method. In particular, stabilization properties of ξ yields for any $(y,h') \in \mathcal{R}_\lambda$ that the second order correlation function

$$\mathbb{E}\left[\lambda^{2\eta[\xi]}\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)} \cup (v',h'))\xi^{(\lambda)}((v',h'),\mathcal{P}^{(\lambda)}) \cup (\mathbf{0},h))\right]$$

converges as $\lambda \to \infty$ to

$$\mathbb{E}\left[\xi^{(\infty)}((\mathbf{0},h),\mathcal{P} \cup (v',h'))\xi^{(\infty)}((v',h'),\mathcal{P} \cup (\mathbf{0},h))\right],$$

the analog of Lemma 3.4 of [40]. As signalled above, we refer the reader to [40] for complete considerations in the particular but very representative case of ξ being the vertex-counting functional.

We shall refer to the statements given in Theorem 7.1 as the *measure-level variance asymptotics and CLT* for $\lambda^{\eta[\xi]}\mu_\lambda^\xi$ with scaling exponent $\tau/2$ and with variance density σ_ξ^2. Theorem 7.1 admits a multivariate version giving a CLT for the random vector $[\lambda^{-\tau/2}\langle g_1, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle, ..., \lambda^{-\tau/2}\langle g_m, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle]$, which follows from the Cramér-Wold device. The question whether $\sigma^2(\xi^\infty) > 0$ is non-trivial and the application of general techniques of stabilization theory designed to check this condition may be far from straightforward. Fortunately, the variances $\sigma^2(\xi_r^{(\infty)}), \sigma^2(\xi_s^{(\infty)})$ and $\sigma^2(\xi_{\vartheta_k}^{(\infty)}),\ k \in \{1,\dots,d-1\}$, admit alternative expressions enjoying monotonicity properties in the underlying Poisson input process $\mathcal{P}$, enabling us to use suitable positive correlation inequalities and to conclude the required positivity for variance densities. More precisely, we have:

Lemma 7.2 *We have*

$$\sigma_s^2 := \sigma^2(\xi_s^{(\infty)}) = \int_{\mathbb{R}^{d-1}} \mathrm{Cov}(\partial\Psi(0), \partial\Psi(v))dv,$$

$$\sigma_r^2 := \sigma^2(\xi_r^{(\infty)}) = \int_{\mathbb{R}^{d-1}} \mathrm{Cov}(\partial\Phi(0), \partial\Phi(v))dv$$

and

$$\sigma_k^2 := \sigma^2(\xi_{\vartheta_k}^{(\infty)}) = \int_{\mathbb{R}^{d-1}} \mathrm{Cov}\left(\int_0^{\Phi(0)} \vartheta_k^\infty((0,h))dh, \int_0^{\Phi(v)} \vartheta_k^\infty((v,h))dh\right) dv.$$

Proof. We only consider the functional $\xi_s^{(\infty)}$, the remaining cases being analogous. For $x = (v,h) \in \mathrm{ext}(\Psi) = \mathrm{Vertices}(\Phi)$ denote by $V[x] = V[x;\mathcal{P}]$ the set of all $v' \in \mathbb{R}^{d-1}$ for which there exists h' with $(v',h') \in \mathcal{F}^\infty(x,\mathcal{P})$ – in other words, $V[x]$ is the spatial projection of all faces f of Φ with $x = \mathrm{Top}(f)$. Clearly, $\{V[x],\ x \in \mathrm{ext}(\Psi)\}$ forms a tessellation of $\mathbb{R}^{d-1}$. Since $\xi_s^{(\infty)}$ is itself an exponentially stabilizing functional on Poisson input on $\mathbb{R}^{d-1} \times \mathbb{R}_+$ (recall Lemma 6.1), it follows from consideration of the second order correlation functions for $\xi^{(\infty)}$ at (7.4) that $\sigma^2(\xi_s^{(\infty)})$ is the asymptotic variance density for $\xi_s^{(\infty)}$, that is to say

$$\sigma^2(\xi_s^{(\infty)}) = \lim_{T\to\infty} \frac{1}{T^{d-1}} \mathrm{Var}\left(\sum_{x=(v,h)\in\mathcal{P},\ v\in[0,T]^{d-1}} \xi_s^{(\infty)}(x;\mathcal{P}) \right).$$

Thus, by definition of $\xi_s^{(\infty)}$,

$$\sigma^2(\xi_s^{(\infty)}) = \lim_{T\to\infty} \frac{1}{T^{d-1}} \mathrm{Var}\left(\sum_{x=(v,h)\in\mathrm{Vertices}(\Psi),\ v\in[0,T]^{d-1}} \int_{V[x]} \partial\Psi(u)du \right).$$

Consequently,

$$\sigma^2(\xi_s^{(\infty)}) = \lim_{T\to\infty} \frac{1}{T^{d-1}} \mathrm{Var}\left(\int_{[0,T]^{d-1}} \partial\Psi(u)du \right) = \lim_{T\to\infty} \frac{1}{T^{d-1}} \int_{([0,T]^{d-1})^2} \mathrm{Cov}(\partial\Psi(u), \partial\Psi(u'))du'du,$$

where the existence of the integrals follows from the exponential localization of $\xi_s^{(\infty)}$, as stated in Lemma 6.1, implying the exponential decay of correlations. Further, by stationarity of the process $\partial\Psi(\cdot)$, the above equals

$$\lim_{T\to\infty} \frac{1}{T^{d-1}} \int_{[0,T]^{d-1}} \int_{[0,T]^{d-1}} \mathrm{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(u'-u))du'du =$$

$$\lim_{T\to\infty} \int_{[-T,T]^{d-1}} \frac{\mathrm{Vol}([0,T]^{d-1}\cap([0,T]^{d-1}+w))}{T^{d-1}} \mathrm{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(w))dw =$$

$$\int_{\mathbb{R}^{d-1}} \mathrm{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(w))dw$$

as required, with the penultimate equality following again by exponential localization of $\xi_s^{(\infty)}$ implying the exponential decay of correlations and thus allowing us to apply dominated convergence theorem to determine the limit of integrals. This completes the proof of Lemma 7.2. $\square$

Observe that, for each v, $\partial\Psi(v)$, $\partial\Phi(v)$ as well as $\int_0^{\Phi(v)} \vartheta_k^{\infty}((v,h))dh$ are all non-increasing functionals of $\mathcal{P}$ and therefore

$$\mathrm{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(v)) \geq 0,\ \mathrm{Cov}(\partial\Phi(\mathbf{0}), \partial\Phi(v)) \geq 0 \text{ and}$$
$$\mathrm{Cov}\left(\int_0^{\Phi(\mathbf{0})} \vartheta_k^{\infty}((\mathbf{0},h))dh, \int_0^{\Phi(v)} \vartheta_k^{\infty}((v,h))dh \right) \geq 0$$

for all $v \in \mathbb{R}^{d-1}$ in view of the positive correlations property of Poisson point processes, see Proposition 5.31 in [34]. It is also readily seen that these covariances are not identically zero, because for $v = 0$ they are just variances of non-constant random variables and, depending continuosly on v, they are strictly positive on a non-zero measure set of v's.Thus, the integrals in variance expressions given in Lemma 7.2 are all strictly positive. Consequently, we have

Corollary 7.1 *The variance densities* $\sigma^2(\xi_r^{(\infty)}), \sigma^2(\xi_s^{(\infty)})$ *and* $\sigma^2(\xi_{\vartheta_k}^{(\infty)}),\ k \in \{1, \ldots, d-1\}$ *are all strictly positive.*

Note that, for $\delta = 0$, the variance positivity for $\sigma^2(\xi_{\vartheta_k}^{(\infty)})$ has been established in a slightly different but presumably equivalent context (binomial input) in [3], Theorem 1.

We also have good reasons to believe that the variance density $\sigma^2(\xi_{f_k}^{(\infty)})$ is strictly positive as well – this is because of the asymptotic non-degeneracy of the corresponding so-called add-one cost functional [6, 25, 27, 28, 30]. Making this intuition precise seems to require additional technical considerations though, which we postpone for future work.Again, for the particular case $\delta = 0$ the required variance positivity is known to hold, as shown by Reitzner in his important work [33].

Let $d\kappa_d$ be the total surface measure of $\mathbb{S}_{d-1}$. Using (6.2,6.4) and (6.5), applying Corollary 7.1 and Theorem 7.1 with $g \equiv 1$, and using $\lambda^{-\tau}\lambda^{2\eta[\xi_s]} = \lambda^{(d+3)/(d+1+2\delta)}$, and likewise for $\xi_r^{(\infty)}$ and $\xi_{\vartheta_k}^{(\infty)}$, we have:

Theorem 7.2 *The random variables* W_λ, V_λ *and* $V_k(K_\lambda),\ k \in \{1, ..., d-1\}$*, satisfy the scalar variance asymptotics and CLT with scaling exponent* $\zeta/2$ *and with strictly positive variance* $\sigma_W^2 := \sigma^2(\xi_s^{(\infty)})d\kappa_d$*,* $\sigma_V^2 := \sigma^2(\xi_r^{(\infty)})d\kappa_d$ *and* $\sigma_{V_k}^2 := \sigma^2(\xi_{\vartheta_k}^{(\infty)})d\kappa_d$ *respectively, where* $\zeta := (d+3)/(d+1+2\delta)$ *is as in (2.32).*

Remark. Recalling (2.33) and setting $\delta = 0$, we obtain from Theorem 7.2 the advertised variance limits (1.1), (1.2), and (1.4). We will show much more for the $\mathbb{R}^{d-1}$-indexed processes $W_\lambda(\cdot)$ and $V_\lambda(\cdot)$ in Section 8.

Next, using (6.7) and Theorem 7.1 we obtain:

Theorem 7.3 *For each* $k \in \{0, \ldots, d-1\}$*, the k-face empirical measures* $\mu_\lambda^{f_k}$ *satisfy the measure-level variance asymptotics and CLT with scaling exponent* $\tau/2$ *and with variance density* $\sigma^2(\xi_{f_k}^{(\infty)})$ *where* $\tau := (d-1)/(d+1+2\delta)$*. In particular, the total number* $f_k(K_\lambda)$ *of k-faces for* K_λ *satisfies the scalar variance asymptotics and CLT with scaling exponent* $\tau/2$ *and variance* $\sigma_{f_k}^2 := \sigma^2(\xi_{f_k}^{(\infty)})d\kappa_d$*.*

Remark. Setting $\delta = 0$ in Theorem 7.3 gives the advertised variance limit (1.3).

8 Global regime and Brownian limits

In this section we establish a functional central limit theorem for the integrated convex hull functions $\hat{W}_\lambda$ and $\hat{V}_\lambda$, defined at (2.33). The methods extend to yield functional central limit theorems for stabilizing functionals in general, thus extending [41].

For any $\sigma^2 > 0$ let B^{σ^2} be the Brownian sheet of variance coefficient σ^2 on the injectivity region $\mathbb{B}_{d-1}(\pi)$ of $\exp := \exp_{\mathbb{S}_{d-1}}$, that is to say B^{σ^2} is the mean zero continuous path Gaussian process indexed by $\mathbb{R}^{d-1}$ with

$$\mathrm{Cov}(B^{\sigma^2}(v), B^{\sigma^2}(w)) = \sigma^2 \cdot \sigma_{d-1}(\exp([\mathbf{0}, v] \cap [\mathbf{0}, w]))$$

where, recall, σ_{d-1} is the $(d-1)$-dimensional surface measure on $\mathbb{S}_{d-1}$. Even though B^{σ^2} is formally indexed by the whole of $\mathbb{R}^{d-1}$, we a.s. have $B^{\sigma^2}(v) = B^{\sigma^2}(w)$ as soon as $[\mathbf{0}, v] \cap \mathbb{B}_{d-1}(\pi) = [\mathbf{0}, w] \cap \mathbb{B}_{d-1}(\pi)$. Recalling from Lemma 7.2 the shorthand notation $\sigma_s^2 := \sigma^2(\xi_s^{(\infty)}), \sigma_r^2 := \sigma^2(\xi_r^{(\infty)})$ and $\sigma_k^2 := \sigma^2(\xi_{\vartheta_k}^{(\infty)})$, the main result of this section is:

Theorem 8.1 *As $\lambda \to \infty$, the random functions $\hat{W}_\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ converge in law to $B^{\sigma_s^2}$ in the space $\mathcal{C}(\mathbb{R}^{d-1})$. Likewise, the random functions $\hat{V}_\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ converge in law to $B^{\sigma_r^2}$ in $\mathcal{C}(\mathbb{R}^{d-1})$.*

Proof. Our argument relies heavily on the theory developed in [40] and further extended in Section 6. For $v \in \mathbb{R}^{d-1}$ and $x \in \mathbb{B}_d$ define

$$\mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}(x) := \begin{cases} 1, & \text{if } x/|x| \in \exp([\mathbf{0}, v]), \\ 0, & \text{otherwise.} \end{cases} \tag{8.1}$$

Using the relations (6.1,6.3) and the identities

$$H_\lambda^{\xi_s}(v) = \langle \mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}, \mu_\lambda^{\xi_s} \rangle, \;\; H_\lambda^{\xi_r}(v) = \langle \mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}, \mu_\lambda^{\xi_r} \rangle,$$

we immediately see that

$$W_\lambda(v) \sim \langle \mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}, \mu_\lambda^{\xi_s} \rangle, \;\; V_\lambda(v) \sim \langle \mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}, \mu_\lambda^{\xi_r} \rangle$$

uniformly in v. Here we use $\sim$ to denote '*asymptotic to*', which we recall means the relevant expressions differ by negligible lower order terms as $\lambda \to \infty$.

More importantly, by (2.33), we have uniformly in v,

$$\hat{W}_\lambda(v) \sim \lambda^{\zeta/2} \langle \mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}, \bar{\mu}_\lambda^{\xi_s} \rangle, \;\; \hat{V}_\lambda(v) \sim \lambda^{\zeta/2} \langle \mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}, \bar{\mu}_\lambda^{\xi_r} \rangle. \tag{8.2}$$

Even though $\mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}$ is not a continuous function, it is easily seen that the proofs in [40] hold for a.e. continuous functions and, in fact the central limit theorems of [40] hold for all bounded functions. Thus Theorem 7.1 for ξ_s and ξ_r remain valid upon setting the test function g to $\mathbf{1}_{\mathbb{B}_d}^{[\mathbf{0},v]}$. This application of Theorem 7.1, combined with (8.2), yields that the fidis of $(\hat{W}_\lambda(v))_{v \in \mathbb{R}^{d-1}}$

converge to those of $(B^{\sigma^2(\xi_s^{(\infty)})}(v))_{v\in\mathbb{R}^{d-1}}$ and, likewise, the fidis of $(\hat{V}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ converge to those of $(B^{\sigma^2(\xi_r^{(\infty)})}(v))_{v\in\mathbb{R}^{d-1}}$ and also

$$\mathrm{Var}[\hat{W}_\lambda(v)] \to \sigma^2(\xi_s^{(\infty)})(v),$$

with similar variance asymptotics for $\hat{V}_\lambda(v)$, see also Theorems 1.2 and 1.3 in [40]. We claim this can be strengthened to the convergence in law in $\mathcal{C}(\mathbb{R}^{d-1})$. It suffices to establish the tightness of the processes $(\hat{W}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ and $(\hat{V}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$. To this end, we shall focus on $\hat{W}_\lambda$, the argument for $\hat{V}_\lambda$ being analogous, and we will proceed to some extent along the lines of the proof of Theorem 8.2 in [15], which is based on [8]. We extend the definition of W_λ to subsets of $\mathbb{R}^{d-1}$ putting for measurable $B \subseteq \mathbb{R}^{d-1}$

$$W_\lambda(B) := \int_{\exp_{d-1}(B)} s_\lambda(w) d\sigma_{d-1}(w)$$

and letting

$$\hat{W}_\lambda(B) := \lambda^{\zeta/2}(W_\lambda(B) - \mathbb{E}\, W_\lambda(B)). \tag{8.3}$$

It is enough to show

$$\mathbb{E}\left(\hat{W}_\lambda([v,v'])\right)^4 = O(\mathrm{Vol}([v,v'])^2),\ v,v' \in \mathbb{R}^{d-1}, \tag{8.4}$$

for then $\hat{W}_\lambda$ satisfies condition (2) on page 1658 of [8], thus belongs to the class $\mathcal{C}(2,4)$ of [8], and is tight in view of Theorem 3 on page 1665 ibidem. To this end, we put

$$W_\lambda^{\#}(B) := \lambda^{\eta[\xi_s]} W_\lambda(B) = \lambda^{\beta(d-1)+\gamma} W_\lambda(B) \tag{8.5}$$

where, recall, $\eta[\xi_s] = \beta(d-1)+\gamma$ is the proper scaling exponent for ξ_s. The crucial point now is that in analogy to the proof of Lemma 5.3 in [6] and similar to (3.24) in the proof of Theorem 1.3 in [40], by a stabilization-based argument all cumulants of $W_\lambda^{\#}([v,w])$ over rectangles $[v,w]$ are at most linear in $\lambda^\tau \mathrm{Vol}([v,w])$ with τ as in (7.8), that is to say, for all $k \geq 1$, we have

$$\left|c^k(W_\lambda^{\#}([v,w]))\right| \leq C_k \lambda^\tau \mathrm{Vol}([v,w]),\ v,w \in \mathbb{R}^{d-1}, \tag{8.6}$$

where c^k stands for the cumulant of order k and where C_k is a constant. Thus, putting (8.3) and (8.5) together, we get from (8.6)

$$\left|c^k(\hat{W}_\lambda([v,w])\right| \leq C_k \lambda^{k[\zeta/2-\eta[\xi_s]]} \lambda^\tau \mathrm{Vol}([v,w]) = \tag{8.7}$$

$$C_k \lambda^{k[\zeta/2-\beta(d-1)-\gamma]} \lambda^{\beta(d-1)} \mathrm{Vol}([v,w]).$$

To proceed, use the identity $\mathbb{E}\,(Y-\mathbb{E}\,Y)^4 = c^4(Y)+3(c^2(Y))^2$ valid for any random variable Y, to conclude from (8.7), recalling the expressions (2.18) and (2.32) for β,γ,ζ, that for $v,w\in\mathbb{R}^{d-1}$

$$\mathbb{E}\left(\hat{W}_\lambda([v,w])\right)^4 = O(\lambda^{4[\zeta/2-\beta(d-1)-\gamma]}\lambda^{\beta(d-1)}\mathrm{Vol}([v,w])) + O([\lambda^{2[\zeta/2-\beta(d-1)-\gamma]}\lambda^{\beta(d-1)}\mathrm{Vol}([v,w])]^2) \tag{8.8}$$

$$= O(\lambda^{-\beta(d-1)}\mathrm{Vol}([v,w])) + O(\mathrm{Vol}([v,w])^2)$$

which is of the required order $O(\mathrm{Vol}([v,w])^2)$ as soon as $\mathrm{Vol}([v,w]) = \Omega(\lambda^{-\beta(d-1)})$. Thus we have shown (8.4) for $\mathrm{Vol}([v,w]) = \Omega(\lambda^{-\beta(d-1)})$ and we have to show it holds for $\mathrm{Vol}([v,w]) = O(\lambda^{-\beta(d-1)})$ as well. To this end, we use that $W_\lambda([v,w]) = \lambda^{-\gamma}O_P(\mathrm{Vol}([v,w]))$ with γ being the height coordinate re-scaling exponent, and $\mathbb{E}\,[W_\lambda([v,w]) - \mathbb{E}\,W_\lambda([v,w])]^4 = \lambda^{-4\gamma}O(\mathrm{Vol}([v,w])^4)$ and thus, by (8.3)

$$\mathbb{E}\left(\hat{W}_\lambda([v,w])\right)^4 = \lambda^{4[\zeta/2]}\lambda^{-4\gamma}O(\mathrm{Vol}([v,w])^4).$$

Recalling (2.18) and (2.32) and using that $\mathrm{Vol}([v,w]) = O(\lambda^{-\beta(d-1)})$ we conclude that

$$\mathbb{E}\left(\hat{W}_\lambda([v,w])\right)^4 = O(\lambda^{2\beta(d-1)}\mathrm{Vol}([v,w])^4) = O(\mathrm{Vol}([v,w]^2))$$

as required, which completes the proof of the required relation (8.4). Having obtained the required tightness we get the convergence in law of $(\hat{W}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ to $(B^{\sigma_s^2}(v))_{v\in\mathbb{R}^{d-1}}$ and, likewise, of $(\hat{V}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ to $(B^{\sigma_r^2}(v))_{v\in\mathbb{R}^{d-1}}$in $\mathcal{C}(\mathbb{R}^{d-1})$ which completes the proof of Theorem 8.1. □

9 Extreme value asymptotics

This section establishes a convergence result for the common extremal value of s_λ and r_λ. The method is based on Janson's result on random coverings of a compact Riemannian manifold by small geodesic balls (Lemma 8.1. in [21]) and consequently works in any dimension. Recalling the definition of γ at (2.18), let $S_\lambda := \sup_{u\in\mathbb{S}_{d-1}} s_\lambda(u)$ and define

$$G_\lambda := M\lambda S_\lambda^{\frac{1}{\gamma}} - C_1\log(\lambda) - C_2\log(\log(\lambda)) - C_3 \tag{9.1}$$

with

$$M := \frac{2^{\frac{d-1}{2}}\Gamma(\delta+1)\Gamma\left(\frac{d+1}{2}\right)\kappa_{d-1}}{\Gamma\left(\frac{2\delta+d+3}{2}\right)},$$

and where C_1, C_2, and C_3 are explicit constants given by (9.6) below.

Theorem 9.1 *As λ tends to ∞, the random variable G_λ defined at (9.1) converges in distribution to the Gumbel-extreme value distribution, i.e. for every $t \in \mathbb{R}$,*

$$\lim_{\lambda\to\infty} P[G_\lambda \le t] = \exp\left(-e^{-t}\right).$$

Remarks. (i) In $d = 2$, Bräker et al. [10] shows that the Hausdorff distance between $\mathbb{B}_2$ and the polytope K_n arising from n i.i.d. uniform points in $\mathbb{B}_2$, after centering and scaling, converges to a Gumbel extreme value distribution.

(ii) For any $d = 2, 3, ...$, Mayer and Molchanov [23] find the limit distribution of the scaled diameter of K_n and K_λ, after centering around 2.

Proof. Fix $t \in \mathbb{R}$ and consider the quantity

$$f(\lambda) := f(\lambda; t) := \left[\frac{1}{M}\left(C_1\frac{\log(\lambda)}{\lambda} + C_2\frac{\log(\log(\lambda))}{\lambda} + \frac{C_3 + t}{\lambda}\right)\right]^\gamma, \tag{9.2}$$

noting that $(G_\lambda \le t)$ if and only if $(S_\lambda \le f(\lambda))$. For each $x \in \mathcal{P}_\lambda$, we let B_x be the ball $B_d(x/2, |x|/2)$. We have $(S_\lambda \le f(\lambda))$ if and only if the sphere $S(\mathbf{0}, (1 - f(\lambda)))$ centered at the origin and of radius $(1 - f(\lambda))$ is fully covered by the spherical patches $B_x \cap S(\mathbf{0}, (1 - f(\lambda)))$, $x \in \mathcal{P}_\lambda$. Only the balls B_x with x in the annulus $\{x; 1 - f(\lambda) \le |x| \le 1\}$ are useful, so we will restrict attention to these points of $\mathcal{P}_\lambda$.

Let us now focus on this covering probability. After a homothetic transformation, it becomes the covering of $\mathbb{S}_{d-1}$ by a set of spherical patches defined as follows. We construct a homogeneous Poisson point process on $\mathbb{S}_{d-1}$ of intensity

$$\Lambda := \Lambda(\lambda) := \lambda\frac{1}{d\kappa_d}\int_{1-f(\lambda)<|x|<1}(1 - |x|)^\delta dx \underset{\lambda\to\infty}{\sim} \lambda\frac{f(\lambda)^{\delta+1}}{\delta + 1}, \tag{9.3}$$

i.e. the mean number of points of $\mathcal{P}_\lambda$ in the annulus $\{x; 1 - f(\lambda) \le |x| \le 1\}$ divided by the area $d\kappa_d$ of $\mathbb{S}_{d-1}$. Around any point x of this point process, we construct independently a spherical patch of geodesic radius $R_\lambda := \arccos\left(\frac{1-f(\lambda)}{Z}\right)$ where Z is a random variable distributed as the norm of a uniform point in the annulus $\{x; 1 - f(\lambda) \le |x| \le 1\}$. In other words, the density of Z is

$$\frac{1}{C_0}\rho^{d-1}(1 - \rho)^\delta \mathbf{1}_{[1-f(\lambda),1]}(\rho),$$

where C_0 is a normalizing constant. In particular, as $\lambda \to \infty$, it can be verified that the normalized geodesic radius R_λ satisfies

$$a_\lambda^{-1} R_\lambda \xrightarrow{D} \mathbf{1}_{[0,1]}(v)2(\delta + 1)(1 - v^2)^\delta v dv, \tag{9.4}$$

where $a_\lambda := \sqrt{2f(\lambda)}$ goes to 0. Indeed, for any measurable function h on $\mathbb{R}_+$, we have

$$\begin{aligned}
\mathbb{E}\left[h(a_\lambda^{-1}R_\lambda)\right] &= \frac{1}{C_0}\int_{1-f(\lambda)}^{1} h\left(a_\lambda^{-1}\arccos\left(\frac{1-f(\lambda)}{\rho}\right)\right)(1-\rho)^\delta\rho^{d-1}d\rho \\
&= \frac{1}{C_0}\int_0^{\frac{\arccos(1-f(\lambda))}{a_\lambda}} h(u)a_\lambda[f(\lambda)-1+\cos(a_\lambda u)]^\delta(1-f(\lambda))^d\frac{\sin(a_\lambda u)}{\cos^{d+1}(a_\lambda u)}du.
\end{aligned}$$

We conclude by taking the limit of the integrand as $\lambda \to \infty$.

We will use a slightly modified version of an original result due to Janson [21], retaining common notation with Lemma 8.1 of [21] as much as possible. For every $\Lambda, \varepsilon > 0$, let $p_{\Lambda,\varepsilon}$ be the probability of covering the unit-sphere $\mathbb{S}_{d-1}$ (of area $d\kappa_d$) with a Poissonian number of mean $d\Lambda\kappa_d$ of independent and uniformly located spherical patches with a radius distributed as $\varepsilon\widetilde{R}_\Lambda$, $\widetilde{R}_\Lambda$ being a bounded random variable for every Λ. If:

- $\widetilde{R}_\Lambda \xrightarrow{\mathcal{D}} R$ as $\Lambda \to \infty$, R a bounded random variable, and
- ε (going to 0) and Λ (going to ∞) are related such that the following convergence occurs:

$$\lim_{\epsilon\to 0,\Lambda\to\infty}\left\{b\varepsilon^{d-1}d\kappa_d\Lambda + \log(b\varepsilon^{d-1}) - (d-1)\log(-\log(b\varepsilon^{d-1})) - \log\alpha\right\} = t \qquad (9.5)$$

  where $b := \frac{\kappa_{d-1}}{d\kappa_d}\mathbb{E}\left[R^{d-1}\right]$ and

$$\alpha := \frac{1}{(d-1)!}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^{d-2}\cdot\frac{(\mathbb{E}\left[R^{d-2}\right])^{d-1}}{(\mathbb{E}\left[R^{d-1}\right])^{d-2}},$$

then the probability $p_{\Lambda,\varepsilon}$ goes to $\exp(-e^{-t})$.

This statement is indeed very close to Lemma 8.1 in [21] rewritten in the case where the Riemannian manifold considered there is the $(d-1)$-dimensional unit-sphere $\mathbb{S}_{d-1}$. The only notable difference is that the radii of the spherical patches depend here on both ε and Λ whereas in Janson's result, they are taken as εR where R is a fixed random variable. We can prove this new variation by following the same lines as in ([21], pages 109-111). In particular, the required convergence [(7.15), (7.20), *ib.*] still holds in this new setting thanks to our first hypothesis about the convergence in distribution of $\widetilde{R}_\lambda$. The values of the constants b and α are deduced from the formulas in ([21], second line of page 109, (9.24)), applied to the case of $\mathbb{S}_{d-1}$.

We then apply this result with the choice of Λ given by (9.3), $\varepsilon := a_\lambda := \sqrt{2f(\lambda)}$ and $a_\lambda\widetilde{R}_\Lambda := R_\lambda$. The limit law of $a_\lambda^{-1}R_\lambda$ is then provided by (9.4) and the calculation of the moments of this limit distribution leads to the formulae

$$b = \frac{\kappa_{d-1}\Gamma(\delta+2)\Gamma\left(\frac{d+1}{2}\right)}{d\kappa_d\Gamma\left(\frac{2\delta+d+3}{2}\right)},$$

and

$$\alpha = \frac{1}{(d-1)!}\frac{\pi^{\frac{d-2}{2}}(\delta+d+1)^{d-2}\Gamma\left(\frac{d-1}{2}\right)^{d-1}}{2^{d-2}\Gamma\left(\frac{d}{2}\right)^{d-2}}\cdot\frac{\Gamma(\delta+2)}{\Gamma\left(\frac{\delta+d+1}{2}\right)}.$$

Using $\varepsilon = \sqrt{2f(\lambda)}$, (9.3) and (9.2), we observe that when λ goes to infinity,

$$\varepsilon^{d-1}\Lambda = \frac{C_1}{M}\log(\lambda) + \frac{C_2}{M}\log(\log(\lambda)) + \frac{C_3+t}{M} + o(1)$$

so the condition (9.5) occurs as soon as C_1, C_2, and C_3 are adjusted accordingly, i.e.

$$C_1 := \frac{(d-1)\gamma}{2}, \quad C_2 := \frac{(d-1)(2-\gamma)}{2} \text{ and } C_3 := \log(\alpha) - \log\left(\frac{b2^{\frac{d-1}{2}}(C_1/M)^{\frac{\gamma(d-1)}{2}}}{\left(\frac{d-1}{2}\right)^{d-1}\gamma^{d-1}}\right). \quad (9.6)$$

The proof of Theorem 9.1 is now complete. □

Remark. The rewriting of the distribution function of S_λ as a covering probability of the sphere leads to an explicit formula in dimension two with the use of [42]. Besides, denoting by m_λ the infimum of s_λ, we observe that the distribution of m_λ is easy to derive and that we could have obtained similar extreme value-type results for the law of S_λ conditioned on $\{m_\lambda = t\}$, $t > 0$.

10 Dual results for zero-cells of isotropic hyperplane tessellations

This section extends the preceding results to a different dual model of random convex polyhedra. Considering a Poisson point process $\widetilde{P}^{(\lambda)}$ of intensity measure $\lambda\mathbf{1}_{\mathbb{R}^d\setminus\mathbb{B}_d}|x|^{\alpha-d}dx,\ \lambda > 0,\ \alpha \geq 1$, we construct the associated hyperplane process and tessellation of the space (see for instance [38], Section 10.3). The object of interest is the *zero-cell of this tessellation*, denoted by $\mathcal{C}_{\alpha,\lambda}$. Indeed, it can be verified (see [14], Section 1) that $t\mathcal{C}_{d,(2t)^d}$ (respectively $t\mathcal{C}_{1,t}$) is equal in distribution to the typical Poisson–Voronoi cell $\mathcal{C}^{PV,t}$ (respectively the Crofton cell $\mathcal{C}^{Cr,t}$) conditioned on having its inradius greater than $t > 0$. The term inradius denotes here the radius of the largest ball centered at the origin and contained in the cell. For sake of simplicity, the dependency on α of the zero-cell $\mathcal{C}_\lambda$ will be omitted. The results presented in this section will hold for any $\alpha \geq 1$ but we will emphasize the two particular cases of the typical Poisson–Voronoi and Crofton cells.

A famous conjecture due to D. G. Kendall (see e.g. the foreword of [43]) states that cells of large area in an isotropic Poisson line tessellation are close to the circular shape. Hug, Reitzner and Schneider [20] have proved and extended this fact to a very general model of zero-cell of a hyperplane tessellation. When the inradius of the cell is large, we are in a particular case of the conjecture so we expect the cell to converge to the spherical shape. In previous works [13, 14], we have obtained more specific and quantitative information such as the behavior of the circumscribed radius or the growth of the number of hyperfaces. In the present paper, we aim at deriving properties similar to those presented in the previous sections for random polytopes in the unit-ball. On a side note, it may be interesting to remark that asymptotically parabolic-shaped faces have already appeared in the study of random tessellations. Indeed, when looking at the planar typical Poisson-Voronoi cell with many sides, Hilhorst observes that the boundary separating the set of first from the set of second-neighbor cells is piecewise parabolic [18].

Limit results for the typical Poisson–Voronoi and Crofton cells may be obtained either by directly using the techniques of the preceding sections or, more efficiently, as consequences of the already existing limit results for random polytopes in $\mathbb{B}_d$. Indeed, let us consider the inversion function I on $\mathbb{R}^d \setminus \{\mathbf{0}\}$, i.e. $I(x) = x/|x|^2$ for every $x \neq \mathbf{0}$. Then I sends $\mathbb{R}^d \setminus \mathcal{C}_\lambda$ to the Voronoi flower $F(K_\lambda)$ of the random polytope K_λ in $\mathbb{B}_d$ associated with a Poisson point process of intensity measure $\lambda|x|^{-\alpha-d}dx$, $x \in \mathbb{B}_d$. This intensity measure converges to λdx as $|x| \to 1$, so that characteristics of K_λ, and hence those of $\mathcal{C}_\lambda$, can be treated via the general methods established in Sections 3-8 with the particular choice $\delta = 0$ and by noticing that the results of these sections hold for the intensity measure $\lambda|x|^{-\alpha-d}dx$.

To see this, let $\widetilde{r}_\lambda$ be the defect radius-vector function of $\mathcal{C}_\lambda$, i.e. for all $u \in \mathbb{S}_{d-1}$,

$$\widetilde{r}_\lambda(u) := \sup\{\rho > 0, \rho u \in \mathcal{C}_\lambda\} - 1.$$

If s_λ denotes the defect support function of K_λ then the following relation is satisfied:

$$s_\lambda = 1 - \frac{1}{1 + \widetilde{r}_\lambda}. \tag{10.1}$$

Since $s_\lambda \simeq \widetilde{r}_\lambda$ for large λ, the asymptotic behaviors of s_λ and $\widetilde{r}_\lambda$ coincide. Letting $\delta = 0$ in Theorem 4.1 yields that for any $R > 0$, as $\lambda \to \infty$, the random function $u \longmapsto \lambda^{\frac{2}{d+1}} \widetilde{r}_\lambda(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} u))$ converge in law to $\partial\Psi$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$. Additionally, in dimension two, the fidis of $u \longmapsto \lambda^{2/3}\widetilde{r}_\lambda(\lambda^{-1/3}u)$ coincide with those given in Proposition 5.1.

In the next theorem, we rewrite the convergence of $\widetilde{r}$ in the particular cases of the typical

Poisson–Voronoi cell (i.e. by taking $\alpha = d$, $\lambda = (2t)^d$ and then applying a scale factor of t) and of the Crofton cell (i.e. by taking $\alpha = 1$, $\lambda = t$ and then applying a scale factor of t).

For $u \in \mathbb{S}_{d-1}$, recall that the *radius-vector function* of a set K containing the origin in the direction of u is given by $r_K(u) := \sup\{\varrho > 0,\ \varrho u \in K\}$. Unlike $\widetilde{r}_\lambda$, it is not centered.We have:

Theorem 10.1 *As $\lambda \to \infty$, both*

$$\left(4^{\frac{d}{d+1}} t^{\frac{d-1}{d+1}} \left[r_{\mathcal{C}^{PV,t}}(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} u)) - t\right]\right)_{u \in \mathbb{R}^{d-1}} \text{ and } \left(t^{-\frac{d-1}{d+1}} \left[r_{\mathcal{C}^{Cr,t}}(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} u)) - t\right]\right)_{u \in \mathbb{R}^{d-1}}$$

converge in law to $\partial\Psi$ in $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ for any $R > 0$.

Direct consequences include the vague convergence of the k-face empirical measures and of the curvature measures to their equivalents for the parabolic growth process. To further clarify the duality between the two models of random convex polyhedra, notice that the set of k-faces of $\mathcal{C}_\lambda$ is in bijection with the set of $(d-k-1)$-faces of K_λ for every $0 \le k \le (d-1)$. Indeed, a k-face of $\mathcal{C}_\lambda$ is the non-empty intersection of $\partial\mathcal{C}_\lambda$ with $(d-k)$ hyperplanes from the original hyperplane process. It is sent by the inversion function to the non-empty intersection of $\partial F(K_\lambda)$ with $(d-k)$ balls $B(x_1/2, |x_1|/2)$,..., $B(x_{d-k}/2, |x_{d-k}|/2)$ where $x_1, ..., x_{d-k}$ are extreme points from the underlying Poisson point process in $\mathbb{B}_d$. It remains to observe that the intersection of $(d-k)$ such balls meets the boundary of the flower if and only if there is a support hyperplane containing $\mathrm{aff}[x_1, \ldots, x_{d-k}]$, i.e. $\mathrm{conv}(x_1, \ldots, x_{d-k})$ is a $(d-k-1)$-face of K_λ. As a consequence, the total numbers of $(d-k-1)$-faces of $\mathcal{C}_\lambda$ satisfy the scalar variance asymptotics and CLT as given in Theorem 7.3.

Theorem 9.1 can be extended as well. Denote by $\widetilde{S}_\lambda$ the supremum of $\widetilde{r}_\lambda$ and recall that S_λ is the supremum of s_λ as in Section 9. Using the relation (10.1), we have $\widetilde{S}_\lambda = \frac{1}{1-S_\lambda} - 1 = S_\lambda(1 + S_\lambda + S_\lambda^2 + ...)$, whence $|\widetilde{S}_\lambda^{1/\gamma} - S_\lambda^{1/\gamma}| = O(S_\lambda^{1+1/\gamma})$ and so by Theorem 9.1 we have $\lambda(\widetilde{S}_\lambda^{1/\gamma} - S_\lambda^{1/\gamma}) \to 0$ in probability as $\lambda \to \infty$. Consequently, Theorem 9.1 (with the particular choice $\delta = 0$ and $\gamma = \frac{2}{d+1}$) holds when S_λ is replaced with $\widetilde{S}_\lambda$:

Theorem 10.2 *Let $\mathcal{C}^{PV,t}$ (respectively $\mathcal{C}^{Cr,t}$) be the typical Poisson-Voronoi cell (respectively the Crofton cell) conditioned on having its inradius greater than $t > 0$. Let $R_{PV,t}$ (respectively $R_{Cr,t}$) be the radius of the smallest ball centered at the origin and containing $\mathcal{C}^{PV,t}$ (respectively $\mathcal{C}^{Cr,t}$). Then*

$$\frac{2^{\frac{3d+1}{2}} \kappa_{d-1}}{d+1} t^{\frac{d-1}{2}} (R_{PV,t} - t)^{\frac{d+1}{2}} - C_1' \log(t) - C_2' \log(\log(t)) - C_3$$

converges to the Gumbel law, where $C_1' = dC_1$, $C_2' = C_2$ *and* $C_3' = C_3 + d\log(2)C_1 + C_2\log(d)$, C_1, C_2 *and* C_3 *being given by (9.6). Likewise*

$$\frac{2^{\frac{d+1}{2}}\kappa_{d-1}}{d+1}t^{-\frac{d-1}{2}}(R_{Cr,t}-t)^{\frac{d+1}{2}} - C_1\log(t) - C_2\log(t\log(t)) - C_3$$

converges to the Gumbel law.

Remark. Theorem 10.2 extends previous results of [13] obtained on the circumscribed radius of both these cells in dimension two.

Similarly, if we denote by $\widetilde{W}_\lambda(v) := \int_{\exp([\mathbf{0},v])}\widetilde{r}_\lambda(w)d\sigma_{d-1}(w)$, $v \in \mathbb{R}^{d-1}$, then the relation (10.1) combined with the previous estimation of the supremum of $\widetilde{r}_\lambda$ implies that $\lambda^{\frac{d+3}{2(d+1)}}[\widetilde{W}_\lambda(v) - W_\lambda(v)]$ converges to 0 almost surely and in L^1, uniformly in v. Consequently, by Theorem 8.1, $\left(\lambda^{\frac{d+3}{2(d+1)}}(\widetilde{W}_\lambda(u) - \mathbb{E}\,\widetilde{W}_\lambda(u))\right)_{u\in\mathbb{R}^{d-1}}$ converges in law to $B^{\sigma^2(\xi_s^{(\infty)})}$ in the space $\mathcal{C}(\mathbb{R}^{d-1})$:

Theorem 10.3 *Let* $\mathcal{C}^{PV,t}$ *be the typical Poisson-Voronoi cell conditioned on having its inradius greater than* $t > 0$. *For every* $v \in \mathbb{R}^d$, *we define* $\widetilde{W}_{PV,t}(v) := \int_{\exp_{d-1}([\mathbf{0},v])}(\widetilde{r}_{PV,t}(w) - t)d\sigma_{d-1}(w)$. *Then the random function* $u \longmapsto 2^{\frac{d+3}{2(d+1)}}t^{-\frac{d-1}{2(d+1)}}[\widetilde{W}_{PV,t}(u) - \mathbb{E}\,\widetilde{W}_{PV,t}(u)]$ *converges in law to* $B^{\sigma^2(\xi_s^{(\infty)})}$ *in the space* $\mathcal{C}(\mathbb{R}^{d-1})$. *Likewise, if* $\mathcal{C}^{Cr,t}$ *is the typical Crofton cell conditioned on having its inradius greater than* $t > 0$ *and if* $\widetilde{W}_{Cr,t}(v) := \int_{\exp_{d-1}([\mathbf{0},v])}(\widetilde{r}_{Cr,t}(w) - t)d\sigma_{d-1}(w)$), *then* $u \longmapsto t^{-\frac{d-1}{2(d+1)}}[\widetilde{W}_{Cr,t}(u) - \mathbb{E}\,\widetilde{W}_{Cr,t}(u)]$) *converges in law to* $B^{\sigma^2(\xi_s^{(\infty)})}$ *in the space* $\mathcal{C}(\mathbb{R}^{d-1})$.

Remark. In dimension $d = 2$, it is shown in [14] that the cumulative value

$$\widetilde{W}_{PV,t}(v) := \int_{\mathbb{S}_{d-1}}(\widetilde{r}_{PV,t}(w) - t)d\sigma_{d-1}(w)$$

is asymptotically normal. To the best of our knowledge, Theorem 10.3 is the first Brownian limit result for Poisson-Voronoi and Crofton cells.

Appendix

Second-order results for the point process of extremal points

The following results are a continuation of those in Section 5. We focus on the point process $\text{ext}(\mathcal{P}_\lambda)$ of extremal points of $\mathcal{P}_\lambda$. The first proposition will provide the distribution of a typical pair of neighboring extremal points for a fixed λ. Either by taking the limit when λ goes to infinity or by using the parabolic growth process, we will derive a similar distribution in the asymptotic regime. In the second proposition, we obtain an explicit formula for the pair correlation function of the point process of extremal points.

Denote by $x := (\theta, h) \in [0, 2\pi) \times [0, 1]$ the polar coordinates of points x of $\mathcal{P}_\lambda$, where $h := 1 - |x|$. We define A_λ as the set of couples of points (r_1, θ_1) and (r_2, θ_2) ($\theta_1 < \theta_2$) of $\mathcal{P}_\lambda$ which are both extremal and neighbors (i.e. there is no extremal point with an angular coordinate in (θ_1, θ_2)). The typical pair of neighboring extremal points is defined in law as a random variable $(\Theta^{(\lambda)}, H_1^{(\lambda)}, H_2^{(\lambda)})$ with values in $[0, 2\pi) \times [0, 1]^2$ where $\Theta^{(\lambda)}$ is the angular distance between the two points and $H_1^{(\lambda)}, H_2^{(\lambda)}$ are their radial coordinates. The distribution of this typical pair is provided by a Palm-type formula: For any non-negative measurable function $g : \mathbb{R}_+ \times [0, 1]^2 \longrightarrow \mathbb{R}_+$,

$$\mathbb{E}\,[g(\Theta^{(\lambda)}, H_1^{(\lambda)}, H_2^{(\lambda)})] = \frac{1}{c}\mathbb{E}\left[\sum_{(\theta_1,h_1),(\theta_2,h_2)\in\mathcal{P}_\lambda} g(|\theta_2 - \theta_1|, h_1, h_2)\mathbf{1}_{A_\lambda}((\theta_1, h_1), (\theta_2, h_2))\right] \tag{A1}$$

where c is a normalizing constant. In the same way, we define the distribution of the typical pair of neighboring extremal points of the parabolic growth process Ψ as a random variable (Θ, H_1, H_2) with values in $\mathbb{R}_+^3$. The formula is very close to (A1), provided that the set A_λ is replaced by the set A of couples of $\mathbb{R} \times \mathbb{R}_+$ such that the two points are extremal points of Ψ and neighbors (i.e. the second point is the right-neighbor of the first).

Proposition *The density of the typical pair of neighboring extremal points* $(\Theta^{(\lambda)}, H_1^{(\lambda)}, H_2^{(\lambda)})$ *of* $\mathcal{P}_\lambda$ *is equal (up to a multiplicative constant) to*

$$\varphi_\lambda(\theta, h_1, h_2) = \exp\left\{-\lambda\left[\arccos(T(\theta, h_1, h_2)) - T(\theta, h_1, h_2)\sqrt{1 - T(\theta, h_1, h_2)^2}\right]\right\}(1 - h_1)(1 - h_2) \tag{A2}$$

where

$$T(\theta, h_1, h_2) = \frac{(1 - h_1)(1 - h_2)\sin(\theta)}{\sqrt{(1 - h_1)^2 + (1 - h_2)^2 - 2(1 - h_1)(1 - h_2)\cos(\theta)}}. \tag{A3}$$

Consequently, the density of the typical pair of neighboring extremal points (Θ, R_1, R_2) *of* $\mathrm{ext}(\Psi)$ *is equal (up to a multiplicative constant) to*

$$\varphi(\theta, h_1, h_2) = \exp\left\{-\frac{4\sqrt{2}}{3}\left[\frac{h_1 + h_2}{2} + \frac{1}{2}\frac{(h_1 - h_2)^2}{\theta^2} + \frac{1}{8}\theta^2\right]^{3/2}\right\}. \tag{A4}$$

Proof. A first remark of interest is that the couple of points (θ_1, h_1), (θ_2, h_2) is in A_λ if and only if the (unique) circular cap containing both (θ_1, h_1) and (θ_2, h_2) on its boundary does not contain any other point of $\mathcal{P}_\lambda$. In the rest of the proof, we denote by $\mathrm{cap}[(\theta_1, h_1), (\theta_2, h_2)]$ this circular cap.

To simplify the formula (A1), we proceed by using the classical Slivnyak formula for Poisson point processes. In particular, we obtain

$$\mathbb{E}\left[\sum_{(\theta_1,h_1),(\theta_2,h_2)\in\mathcal{P}_\lambda} g(|\theta_2-\theta_1|,h_1,h_2)\mathbf{1}_{A_\lambda}(\theta_1,h_1),(\theta_2,h_2))\right]$$

$$= \frac{1}{2}\lambda^2\int_{(0,1)^2\times(0,2\pi)^2} g(|\theta_2-\theta_1|,h_1,h_2)P[\mathrm{cap}[(\theta_1,h_1),(\theta_2,h_2)]\cap\mathcal{P}_\lambda=\emptyset]$$

$$(1-h_1)(1-h_2)dh_1dh_2dr_1dr_2d\theta_1d\theta_2.$$

It remains to calculate the distance $T(\theta,h_1,h_2)$ from the origin to this circular cap (see (A3)) and to apply the formula (5.1) for the area of this cap to get the required result (A2).

We obtain the density of the typical pair of neighboring points of the parabolic growth process by either taking the limit (with the proper re-scaling) as λ goes to infinity of the previous density or by applying the same method to the parabolic growth process. In the latter case, the empty circular cap is replaced by an empty downward parabola. □

Another second-order characteristic traditionally used for describing point processes is the pair correlation function. In the context of the point process $\mathrm{ext}(\Psi)$, it is defined as a function $\rho^{(2)} := \rho^{(2)}_\Psi : (\mathbb{R}\times\mathbb{R}_+)^2 \longrightarrow \mathbb{R}_+$ such that for every non-negative measurable function $g : (\mathbb{R}\times\mathbb{R}_+)^2 \longrightarrow \mathbb{R}_+$, we have

$$\mathbb{E}\left[\sum_{x\neq y\in\mathrm{ext}(\Psi)} g(x,y)\right] = \int g(x,y)\rho^{(2)}(x,y)dxdy.$$

The next proposition establishes a formula for $\rho^{(2)}_\Psi$. We could have obtained an analogous result in the non-asymptotic regime but for sake of simplicity, we only state the result for the parabolic growth process. We use the notation $x \prec y$, $x, y \in \mathbb{R}\times\mathbb{R}_+$, if the first spatial coordinate of x is less than that of y.

Proposition *For every $x, y \in \mathbb{R}\times\mathbb{R}_+$, we have*

$$\begin{aligned}\rho^{(2)}(x,y) &= 2\exp(-\ell(\Pi^{\downarrow}[x,y]))\\ &+\mathbf{1}_{x\prec y}\sum_{n\in\mathbb{N}}\int_{(\mathbb{R}\times\mathbb{R}_+)^n}\mathbf{1}_{x\prec x_1\prec\cdots\prec x_n\prec y}\exp(-\ell(\Pi^{\downarrow}[x,x_1]\cup\Pi^{\downarrow}[x_1,x_2]\cup\cdots\cup\Pi^{\downarrow}[x_n,y]))dx_1\cdots dx_n\\ &+\mathbf{1}_{y\prec x}\sum_{n\in\mathbb{N}}\int_{(\mathbb{R}\times\mathbb{R}_+)^n}\mathbf{1}_{y\prec x_1\prec\cdots\prec x_n\prec x}\exp(-\ell(\Pi^{\downarrow}[y,x_1]\cup\Pi^{\downarrow}[x_1,x_2]\cup\cdots\cup\Pi^{\downarrow}[x_n,x]))dx_1\cdots dx_n.\end{aligned}$$

Remark. The area $\ell(\Pi^{\downarrow}[x,y])$ is given explicitly by formula (A4)).

Proof. Notice that points $x, y \in \mathcal{P}$ are extremal points if and only if there exist a chain $(x = x_0, x_1, \cdots, x_{n+1} = y)$ for a certain (unique) $n \in \mathbb{N}$, such that the points are ordered with respect to the first coordinate and for every $1 \le i \le (n+1)$, x_{i-1} and x_i are extremal neighboring points (i.e. the unique parabola containing these two points on its boundary must be empty). We recall that A is the set of couples of points from $\mathcal{P}$ which are both extremal and neighbors (i.e. the second point is the right-neighbor of the first).

Consequently, if $g : (\mathbb{R} \times \mathbb{R}_+)^2 \longrightarrow \mathbb{R}_+$ is a real bounded measurable function, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{x \neq y \in \mathrm{ext}(\Psi)} g(x,y)\right] &= \mathbb{E}\left[\sum_{x \neq y \in \mathcal{P}} g(x,y)\mathbf{1}_{\mathrm{ext}(\Psi)}(x)\mathbf{1}_{\mathrm{ext}(\Psi)}(y)\right] \\
&= \sum_{n \in \mathbb{N}} \mathbb{E}\left[\sum_{x_0, x_1, \cdots, x_{n+1} \in \mathcal{P}} [g(x_0, x_{n+1}) + g(x_{n+1}, x_0)] \prod_{i=1}^{n+1} \mathbf{1}_A(x_{i-1}, x_i)\right],
\end{aligned}$$

since the chain may start at either x or y. Applying Slivnyak's formula to the last expression, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x \neq y \in \mathrm{ext}(\Psi)} g(x,y)\right] \\
&= \sum_{n \in \mathbb{N}} \int_{(\mathbb{R} \times \mathbb{R}_+)^{(n+2)}} (g(x_0, x_{n+1}) + g(x_{n+1}, x_0)) \\
&\qquad\qquad P[\mathcal{P} \cap \left(\cup_{1 \le i \le (n+1)} \Pi^{\downarrow}[x_{i-1}, x_i]\right) = \emptyset] dx_0 dx_1 \cdots dx_n dx_{n+1} \\
&= \int_{(\mathbb{R} \times \mathbb{R}_+)^2} (g(x,y) + g(y,x)) \Big\{ \exp(-\ell(\Pi^{\downarrow}[x,y])) \\
&\quad + \mathbf{1}_{x \prec y} \sum_{n \in \mathbb{N}} \int_{x \prec x_1 \prec \cdots \prec x_n \prec y} \exp(-\ell(\Pi^{\downarrow}[x, x_1] \cup \Pi^{\downarrow}[x_1, x_2] \cup \cdots \cup \Pi^{\downarrow}[x_n, y])) dx_1 \cdots dx_n \Big\} dxdy.
\end{aligned}$$

The last equality yields the claimed expression for $\rho^{(2)}(x,y)$. □

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